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Top Down Axiomatic Modeling of Metatechnologies and Evaluating Directional Economic Efficiency

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Abstract

We contrast two distinct axiomatic approaches to model metatechnologies. The traditional bottom up approach revolves around axioms characterizing group technologies. In contrast, our innovative top down approach introduces a new set of axioms that directly identify the metatechnology itself. These new axioms lead to discover new algebraic statements of two metatechnologies: in particular, one metatachnology maintaining a new within-group convexity axiom, and another one without it that is essentially nonconvex. Through these metatechnology specific axioms, we derive novel minimum extrapolation results for these two metatechnologies. We adopt a directional economic inefficiency concept as a general measurement framework and its additive decomposition into technical and allocative components. We develop single-stage linear programs for the measurement of directional economic and technical inefficiencies within these two metatechnologies. Our key results are illustrated using a numerical illustration. We manage to establish links between this new top down approach and some key findings from the existing bottom up metatechnology literature.

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1 Introduction

Organizations in different industries, regions and countries may well face different technologies at different points in time. This contribution is concerned with one particular method to account for this type of heterogeneity when estimating production technologies. This problem of accounting for heterogeneity when estimating production relations is actually quite old. One early solution that has likely been initiated by Hayami and Ruttan (1970) involves estimating some type of metaproduction function.[1] We underscore the presence of the idea of an envelope or frontier and note that at least part of this metaproduction function literature allows for some form of inefficiency (e.g., Lau and Yotopoulos (1989)). This metaproduction function concept has been empirically applied in quite a few agricultural studies comparing mainly country-level data (examples include Binswanger et al. (1987) and Lau and Yotopoulos (1989), among others).[2]

The seminal article developing the formal framework for making nonparametric and parametric frontier-based comparisons across groups of firms is commonly attributed to be O'Donnell et al. (2008). This article explicitly builds upon the single output non-frontier metaproduction function approach developed in, e.g., Hayami and Ruttan (1970), and explicitly refers to it (e.g., O'Donnell et al. (2008, footnote 2)). These authors consider a metaproduction technology that is defined as the union of two or more group technologies. The boundary of this metatechnology is referred to as a metafrontier, and the boundaries of the group technologies are called group frontiers. This so-called metatechnology or metafrontier approach has some importance in economic theory (see, e.g., Hung et al. (2009) on optimal growth).[3]

This metafrontier approach has been widely empirically applied across sectors and even disciplines.[4] It also plays a role in designing incentive-compatible managerial policies (e.g., Afsharian (2020)) as well as in evaluating and refining policies of various kinds (e.g., Nguyen et al. (2022), D'Inverno et al. (2021), and Yu et al. (2021)).[5] Meanwhile, this metafrontier framework is extended in various directions: just by way of example we mention the estimation of primal Malmquist and Hicks-Moorsteen productivity indices relative to metafrontiers (see Jin et al. (2020)).

Reliable estimates of a metafrontier allow researchers to determine a variety of performance measures (e.g., technical, economic or scale efficiency, productivity change, capacity utilization). It is common to use assumptions about group technologies to frame the estimation of the metafrontier. Basic group frontier technologies most often make the assumption that the technology is convex. However, even if group technologies are convex, the metatechnology defined by their union is normally not convex (see O'Donnell et al. (2008)).

Despite the mathematical fact that the metatechnology for convex group technologies is generally nonconvex, O'Donnell et al. (2008) estimate the metafrontier as the nonparametric or parametric

[1] To quote at length from this seminal article, Hayami and Ruttan (1970, p. 898) state: "We may call the envelope of all known and potentially discoverable activities a secular or "meta-production function." The full range of technological alternatives described by the meta-production function is only partially available to individual producers in a particular country or agricultural region during any particular historical "epoch." It is, however, potentially available to agricultural scientists and technicians."

[2] An early empirical survey of this almost exclusively agricultural literature is found in Trueblood (1989).

[3] This article explores the complexities of optimal growth when the union of two separate convex technologies yields a basic nonconvexity.

[4] Examples range from agriculture (e.g., Kapelko and Oude Lansink (2020)) to banking (e.g., Casu et al. (2013)), corporate social performance (e.g., Aparicio and Kapelko (2019)), football leagues (e.g., Tiedemann et al. (2011)), hotels (e.g., Huang et al. (2013)), wastewater treatment (e.g., Sala-Garrido et al. (2011)), to name just a few.

[5] A Google Scholar search on 14 March 2024 yields about 12000 results for the search term "metafrontier". In addition, the seminal article of O'Donnell et al. (2008) has obtained 1569 citations on this same date.

boundary of a convex metatechnology. Labeling this potentially wrong estimation strategy as a convexification strategy, Kerstens et al. (2019) theoretically elaborate and empirically illustrate that such a convexification strategy leads to a substantially biased metafrontier within a nonparametric frontier framework. Implications for a proper construction of a nonconvex metafrontier stochastic frontier model have been explored along similar lines in Amsler et al. (2017).[6] In a similar vein, Afsharian (2017) transposes these ideas towards the proper construction of a nonconvex metafrontier stochastic nonparametric model (known as the StoNED method). Though historically only a minority of empirical and methodological articles did not commit this basic erroneous convexification strategy (see, e.g., Sala-Garrido et al. (2011), Tiedemann et al. (2011), and the non-exhaustive list of examples in Kerstens et al. (2019) and Jin et al. (2020)), it is fair to say that this basic error is still widespread among empirical practitioners despite the publication of the articles of Afsharian (2017), Amsler et al. (2017) and Kerstens et al. (2019), among others, pointing out the basic flaws of the convexification strategy.[7]

Composition rules for technologies are rarely discussed in the economic literature (exceptions include Ruys (1974), McFadden (1978), or Chambers et al. (1996)). We are aware of at least three streams of literature that are based on some operation on technologies. First, in the immensely popular metafrontier literature one starts with group technologies representing different production techniques and these different blueprints are compared relative to a metatechnology defined as the *union* of the underlying group technologies. Second, in the aggregation literature on efficiency and productivity one wonders how, e.g., the technical or economic efficiency of individual firms can be meaningfully aggregated at the industry level: answering this question traditionally requires the aggregation of individual firm technologies using the (Minkowski) *sum* operator into an industry technology (see Zelenyuk (2023) for a recent selective review). Third, in the so-called by-production approach initiated by Murty et al. (2012), the by-production technology is conceived as the *intersection* of a conventional technology (that transforms all conventional and polluting inputs into desirable outputs) and an emission-generating technology (that transforms all the polluting inputs into undesirable outputs).

The structure in these three cases is somewhat similar: via some operation on multiple *infra-technologies* a new *supra-technology* is generated. The focus in the literature published thus far has been on the axiomatic foundation of the underlying infra-technologies at the cost of ignoring the exact axioms respected by the resulting supra-technology. The basic question asked in this contribution is whether there is some merit in reversing this perspective. In particular, we focus on the axiomatic foundation of the supra-technology and explore to which extent this differs the axiomatic foundation of the infra-technologies.

In this contribution, we consider the utilization of the union operator in the context of the metafrontier literature and its consequences. The current nonparametric approach to axiomatic modeling of metatechnologies starts by making axioms on the group technologies, but not on the metatechnology itself. This approach differs from the commonly used approach for modeling a vast majority of production technologies in the standard nonparametric production context. This difference creates an ambiguity about the validity of the models obtained based on the current approach. Therefore, this motivates the development of a rigorous modeling approach that is free from any potential ambiguity. Thus, we define our fundamental aim as complementing the existing approach by a –to the best of our knowledge– new approach that starts by directly assuming a series of axioms on the metatechnology solely. While the traditional approach can be labeled as a *bottom up* axiomatic approach,

[6] See, e.g., Tsionas (2020) for a further development accounting for a continuous-time Markov process.

[7] A Google Scholar search on 14 March 2024 reveals that the articles by Afsharian (2017), Amsler et al. (2017) and Kerstens et al. (2019) yield about 22, 43 and 73 citations respectively, while the results for the search term "metafrontier" since 2020 yields 6650 hits. Clearly, the basic message does not transpire in the empirical literature.

the new approach that is being developed in this contribution can probably best be labeled as a *top down* axiomatic approach. Our newly proposed top down approach coincides with the approach used in the standard nonparametric production context in the special case when there is only one group technology.

To establish our top down axiomatic approach, we interpret any metatechnology as a usual technology that transforms inputs to outputs, without imposing any prior link between the metatechnology and its group technologies. Then, we introduce new metatechnology specific production axioms that are used for specifying characteristics of metatechnologies. Our axioms encompass the commonly used axioms in the nonparametric production theory as special cases. One of our introduced axioms is the Inclusion of Group Technologies (IGT) that requires *only* the inclusion of the group technologies in the metatechnology. This axiom weakens an implicit axiom in the bottom up approach of modeling metatechnologies. Namely, the bottom up approach assumes that: (a) the metatechnology includes all group technologies, and (b) the metatechnology is included in the union of group technologies. However, labeling these assumptions together the "Generation by Group Technologies" (GGT) axiom, our axiom IGT incorporates only part (a) of this assumption in our top down approach. In fact, this implies that its part(b) is redundant.

Our newly introduced top down axioms help to formally formulate two types of convexity: within-group convexity (WGC), and between-group convexity (BGC). We axiomatically model two generally nonconvex metatechnologies with and without assuming axiom WGC, respectively: these are called the within-group convexity (WGC), and the nonconvex (NC) metatechnologies. The nonconvex nature of both these metatechnology models may raise the practical question as to which model characterizes the production metatechnology more realistically. In response, our top down approach answers that the nonconvex nature of the WGC metatechnology is due to the absence of the axiom BGC. Analogously, the nonconvexity of the NC metatechnology is due to the absence of the axiom WGC. For the sake of completeness, we also develop a so-called convex pseudo-metatechnology which implements both axioms WGC and BGC. The idea of this development is to clarify that the already mentioned erroneous convexification strategy massively followed in the literature is due to the implicit assumption of axiom BGC, but it is not due to axiom WGC.

While the traditional axioms allow to demonstrate the *minimum extrapolation principle* model selection criterion (see Banker et al. (1984)) for each group technology individually, our new top down axioms enable us to develop new *metatechnology specific* minimum extrapolation results. The importance of these results is apparent by the fact that this guarantees that any axiomatically developed model includes only those units whose feasibility results from the assumed axioms, but not any additional arbitrary units that are not explained by those axioms. To get rid of the ambiguity in developing the WGC and NC metatechnologies in the bottom up approach, we demonstrate that our proposed models are equal to the corresponding existing ones. This establishes the equivalence between the existing bottom up and the new top down axiomatic approaches for modeling metatechnologies.

Our top down axioms also clarify the relation between the returns to scale assumption commonly imposed on each group technology and the returns to scale exhibited by the metatechnology itself. Acknowledging that any real-life metatechnology represents a specific type of returns to scale and finding no empirical evidence that the returns to scale of group technologies may differ, we (theoretically) argue that the returns to scale of a metatechnology is not generally the same one as exhibited by all its group technologies, unless the metatechnology satisfies axiom GGT.

Another point regarding our models of the WGC and NC metatechnologies is that we consider different returns to scale assumptions in their top down development and investigate their relationship.

This leads to extending the existing results established between the four standard technologies with nonincreasing, variable, nondecreasing, and constant returns to scale. Our models of the WGC and NC metatechnologies are valuable because these can be used for the evaluation of global returns to scale (see, e.g., Podinovski (2004)) given the fact that both the WGC and NC metatechnologies are in general nonconvex.

To illustrate the general applicability of our proposed models of the WGC and NC metatechnologies, we focus on the measurement of directional economic efficiency within these metatechnologies. In addition, we discuss the decomposition of this directional economic efficiency into its technical and allocative components. We develop a linear program by which the directional technical inefficiency in the WGC metatechnology can be measured. We prove how this linear program results from formulating the directional distance function of Chambers et al. (1998) based on our novel algebraic model of the WGC metatechnology. We also derive the existing special cases of our proposed program to reveal the implicit ideas behind their development. By demonstrating the identity of the WGC and NC metatechnologies in a special case, we show that other special cases of our proposed linear program are the linear programs developed by Agrell and Tind (2001) and Leleu (2006) for the measurement of Farrell input efficiency in the standard nonconvex technology. It is worth stressing that our proposed approach is a single-stage linear optimization based approach. Therefore, it is computationally more efficient than the existing multi-stage approach that requires solving several linear optimization programs to measure the efficiency of a production unit. By establishing the identity of the NC metatechnology and the standard nonconvex technology (e.g., Deprins et al. (1984)), we find that the enumeration algorithms of Cherchye et al. (2001) and Kerstens and Van de Woestyne (2018), among others, are applicable for measuring directional technical inefficiency in the NC metatechnology. Note that the computational efficiency of these enumeration algorithms is well established in the literature.

Furthermore, exploiting the directional profit inefficiency measure of Sahoo et al. (2014), we introduce a new measure of directional economic inefficiency that encompasses the traditional cost, revenue, and profit inefficiency measures as special cases. The most notable feature of our measure is that it allows any selective subset of inputs and outputs to be incorporated into the measurement of economic inefficiency. From the economic point of view, this enables modeling input fixity, or output fixity, or a combination of both. We additively decompose our proposed economic inefficiency measure into technical and allocative components. To estimate the directional economic inefficiency in the WGC metatechnology, we develop a linear program based on our novel algebraic statement of this metatechnology. Then, we discuss how choosing suitable direction vectors help generally define the directional economic inefficiency and its components in both long-run and short-run contexts. We also follow a similar argument on the evaluation of the directional economic inefficiency relative to the NC metatechnology.

In measuring this directional economic inefficiency in both the WGC and NC metatechnologies, we consider a special case in which the economic inefficiency is evaluated with respect to all inputs and outputs. We prove that a common simple formulation can be applied for the measurement task in these cases. Interestingly, our formulation implies that the maximum profits in both WGC and NC metatechnologies are identical, and some observations can attain this maximum profit. This important result shows that the imposition of axiom WGC does not affect the measurement of directional economic inefficiency in this special case. However, it should be remarked that such equality does not hold when comparing the values of its technical and allocative components.

This contribution starts in the next Section 2 by defining and axiomatically modeling a nonparametric production technology in general, and by articulating the traditional bottom up axiomatic approach to group technologies and the resulting metatechnologies. Section 3 develops in a systematic way our new top down axiomatic approach to modeling a metatechnology with different returns to scale assumptions and with or without convexity. For the sake of methodological clarity, we start our exploration of metatechnologies by adopting axiom WGC in Section 4. Then, we develop some new algebraic statements of the WGC metatechnology that lead to computational efficiency gains. Thereafter, we explore the measurement of directional technical and economic inefficiency with respect to this WGC metatechnology. Then, we move our focus to the NC metatechnology which does not invoke any convexity axiom in Section 5 and we discuss directional technical and economic inefficiency in this framework. Section 6 explicitly considers the convexification strategy by looking at the convexification of the metatechnology. Section 7 presents a numerical illustration for our key directional economic inefficiency decompositions for both the long-run and short-run cases. A concluding Section 8 wraps up the main results and spells out the key implications for practitioners.

2 Production Technology, Group Technology and Metatechnology

2.1 Modeling a Production Technology: An Axiomatic Approach

A production possibilities set or technology $\mathcal{T} \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+$ transforms non-negative input vectors $\mathbf{x} = (x_1, \ldots, x_m)^\top \in \mathbb{R}^m_+$ into non-negative output vectors $\mathbf{y} = (y_1, \ldots, y_s)^\top \in \mathbb{R}^s_+$, and can be conceptually interpreted as follows:[8]

$$\mathcal{T} = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \mathbf{x} \text{ can produce } \mathbf{y}\right\}. \tag{1}$$

Each element $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}$ is referred to as a production unit. In what follows, we consider a set $\mathcal{N}$ of n observations $(\mathbf{x}_j, \mathbf{y}_j)$, $j \in \mathcal{J} = \{1, \ldots, n\}$, such that $\mathbf{x}_j \neq \mathbf{0}$ and $\mathbf{y}_j \neq \mathbf{0}$, for all $j \in \mathcal{J}$. We also assume that for each $i \in \mathcal{I} = \{1, \ldots, m\}$, there is some $j' \in \mathcal{J}$ such that $x_{ij'} > 0$. Additionally, for each $r \in \mathcal{O} = \{1, \ldots, s\}$, there is some $j'' \in \mathcal{J}$ such that $y_{rj''} > 0$.

Since in empirical applications the true production technology is usually unknown, it has to be approximated based on the observed data and a series of assumed hypotheses on the production technology that describe its properties. These hypotheses, stated under the form of production axioms (see Shephard (1970), Banker et al. (1984), or Färe et al. (1985)), are considered such that the approximation becomes as accurate as possible. The more the axioms are well defined and compatible with the underlying technology, the more the chosen model approaches the true model.

An axiomatic approach for non-parametric approximation of technology $\mathcal{T}$ is to consider the intersection of all technologies that satisfy the stated axioms, and verify that the intersection itself satisfies these axioms. Importantly, this verification guarantees that $\mathcal{T}$ only includes those production units that are needed to satisfy the axioms used in its definition and, therefore, does not include any additional arbitrary pairs of input and output vectors that are not explainable by these axioms. Formally stated, technology $\mathcal{T}$ fulfills the following principle used by Banker et al. (1984).

[8] Throughout this paper, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, and $\mathbb{R}^d_+$ (resp., $\mathbb{R}^d_{++}$) denotes its non-negative (resp., strictly positive) orthant. We denote sets by uppercase calligraphic letters and vectors by lowercase boldface letters. We also denote sets whose elements are themselves all sets, by blackboard boldface letters. All vectors are considered to be column vectors and superscript $\top$ denotes transpose. Vectors $\mathbf{0}$ and $\mathbf{1}$ denote vectors of zeroes and ones, respectively. The dimensions of these vectors are clear from the context in which they are used. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, the inequality $\mathbf{a} \geq \mathbf{b}$ ($\mathbf{a} > \mathbf{b}$) means that $a_i \geq b_i$ ($a_i > b_i$), for all $i = 1, \ldots, d$. Furthermore, the notation $\mathbf{a} \otimes \mathbf{b}$ denotes their Hadamard (element-by-element) multiplication.

Definition 2.1. Let $\mathbb{T}$ denote the set of all technologies induced by the given set of observations $\mathcal{N}$ that satisfy a set of axioms. Then, a technology $\mathcal{T}^{\min} \in \mathbb{T}$ satisfies the *minimum extrapolation principle* (MEP) in terms of the stated axioms if $\mathcal{T}^{\min} \subseteq \mathcal{T}$, for all $\mathcal{T} \in \mathbb{T}$.[9]

To illustrate Definition 2.1, consider the following standard axioms defined on all observations:

Axiom IO (*Inclusion of Observations*) $\mathcal{N} \subset \mathcal{T}$.

Axiom SD (*Strong Disposability*) $(\mathcal{T} + \mathcal{C}) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}$, where $\mathcal{C}$ denotes the free disposal cone defined as $\mathcal{C} = \mathbb{R}^m_+ \times -\mathbb{R}^s_+$.

Axiom $\boldsymbol{\Delta}$-RS (Δ-*Returns to Scale*) $\delta\mathcal{T} \subseteq \mathcal{T}$ for all $\delta \in \mathcal{I}_\Delta$, where $\Delta \in \{\mathrm{NI}, \mathrm{V}, \mathrm{ND}, \mathrm{C}\}$ with $\mathcal{I}_{\mathrm{NI}} = \{\delta \in \mathbb{R} \mid 0 \leq \delta \leq 1\}$, $\mathcal{I}_{\mathrm{V}} = \{1\}$, $\mathcal{I}_{\mathrm{ND}} = \{\delta \in \mathbb{R} \mid 1 \leq \delta\}$ and $\mathcal{I}_{\mathrm{C}} = \mathbb{R}_+$.

This axiom imposes a specific assumption regarding the scaling of production units in technology $\mathcal{T}$: Non-Increasing Returns to Scale (NIRS), Variable Returns to Scale (VRS), Non-Decreasing Returns to Scale (NDRS) and Constant Returns to Scale (CRS). Note that a VRS technology satisfies NDRS and NIRS in different regions (see, e.g., Färe et al. (1994)).

Axiom C (*Convexity*) $\mathcal{T}$ is convex.

Denote $\mathbb{T}_{\Delta\text{-C}}$ the set of all technologies $\mathcal{T} \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+$ that satisfy axioms IO, SD, Δ-RS and C. Then, as an extension to both standard convex CRS technology of Charnes et al. (1978) and VRS technology of Banker et al. (1984), the Δ-C technology is defined as $\mathcal{T}^{\min}_{\Delta\text{-C}} = \bigcap_{\mathcal{T} \in \mathbb{T}_{\Delta\text{-C}}} \mathcal{T}$. It is straightforward to verify that technology $\mathcal{T}^{\min}_{\Delta\text{-C}} \in \mathbb{T}_{\Delta\text{-C}}$. Therefore, technology $\mathcal{T}^{\min}_{\Delta\text{-C}}$ satisfies the MEP in terms of the stated axioms. Similarly, as an extension to the standard nonconvex Free Disposal Hull (FDH) technology of Deprins et al. (1984), the Δ-NC technology is defined as $\mathcal{T}^{\min}_{\Delta\text{-NC}} = \bigcap_{\mathcal{T} \in \mathbb{T}_{\Delta\text{-NC}}} \mathcal{T}$, where $\mathbb{T}_{\Delta\text{-NC}}$ denotes the set of all technologies $\mathcal{T} \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+$ that satisfy axioms IO, SD and Δ-RS. It can be also verified that technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ satisfies the MEP in terms of these axioms. Briec et al. (2004) develop a unified statement of technologies $\mathcal{T}^{\min}_{\Delta\text{-C}}$ and $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ as

$$\mathcal{T}^{\min}_{\Delta\text{-}k} = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \sum_{j \in \mathcal{J}} \delta\lambda_j \mathbf{x}_j \leq \mathbf{x},\, \sum_{j \in \mathcal{J}} \delta\lambda_j \mathbf{y}_j \geq \mathbf{y},\, \mathbf{1}^\top \boldsymbol{\lambda} = 1,\, \delta \in \mathcal{I}_\Delta,\, \boldsymbol{\lambda} \in \Lambda_k \Big\}, \tag{2}$$

where $k \in \{\mathrm{C,NC}\}$, $\Lambda_{\mathrm{C}} = \mathbb{R}^n_+$, $\Lambda_{\mathrm{NC}} = \{0,1\}^n$, and $\mathcal{I}_\Delta$ is as considered in axiom Δ-RS. Note that the variable δ represents the scaling factor.

It is worth noting that the Δ-C technology convexifies the Δ-NC technology. Precisely stated, the former technology coincides with the convex hull of the latter one:

$$\mathcal{T}^{\min}_{\Delta\text{-C}} = \mathrm{conv}\left(\mathcal{T}^{\min}_{\Delta\text{-NC}}\right). \tag{3}$$

2.2 Group Technologies and Metatechnology: Original Definitions

Let the set $\mathcal{N}$ be partitioned into G mutually disjoint non-empty subgroups $\mathcal{N}_g$, $g \in \mathcal{G} = \{1, \ldots, G\}$, such that observations in the same subgroup operate in similar conditions. In other words, observations in the same subgroup employ the same *group technology* that is potentially different from the one used

[9] Since any empirical approximation of the "true technology" is referred to as a "technology", the MEP may appear as a property of the true technology. However, we note that the MEP is just a *model selection criterion* by which the smallest (minimal) model of the true technology is chosen among all models of the true technology satisfying a common set of axioms.

in other subgroups. In the remainder, we assume that there is at least two or more subgroups ($G > 1$), unless explicitly stated otherwise. For each $g \in \mathcal{G}$, we denote $\mathcal{J}_g$ the index set of observations in $\mathcal{N}_g$, and denote $\mathcal{T}^g$ the group technology induced by $\mathcal{N}_g$.

For the set of group technologies $\mathcal{T}^g$, $g \in \mathcal{G}$, *metatechnology* $\mathcal{T}^{\mathcal{G}}$ is traditionally interpreted in the literature as follows (see, e.g., Jin et al. (2020)):

$$\mathcal{T}^{\mathcal{G}} = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \mathbf{x} \text{ with some group technology } \mathcal{T}^g, g \in \mathcal{G}, \text{ can produce } \mathbf{y} \right\}. \tag{4}$$

This definition is based on the implicit assumption that, $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{\mathcal{G}}$ if and only if there exists at least one $g' \in \mathcal{G}$ such that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{g'}$. It turns out that $\bigcup_{g \in \mathcal{G}} \mathcal{T}^g \subseteq \mathcal{T}^{\mathcal{G}}$ and $\mathcal{T}^{\mathcal{G}} \subseteq \bigcup_{g \in \mathcal{G}} \mathcal{T}^g$. Therefore, the interpretation (4) of metatechnology $\mathcal{T}^{\mathcal{G}}$ is based on the *prior* assumption that it satisfies the following axiom.

Axiom GGT (*Generation by Group Technologies*) Metatechnology $\mathcal{T}^{\mathcal{G}}$ is equal to the union of its group technologies $\mathcal{T}^g$, $g \in \mathcal{G}$. That is, $\mathcal{T}^{\mathcal{G}} = \bigcup_{g \in \mathcal{G}} \mathcal{T}^g$.

It is worth noting that, even if all group technologies $\mathcal{T}^g$, $g \in \mathcal{G}$, are convex, their union is generally nonconvex, unless there is only one subgroup (i.e., $G = 1$) or otherwise (i.e., $G > 1$) all group technologies are subsets of one of them.

3 Metatechnology: New Definition and Characterization Axioms

The commonly used approach for modeling a vast majority of production technologies in the standard nonparametric production context is the one described in Subsection 2.1. However, looking at the existing models of metatechnologies, it appears that their development does not follow this approach. This naturally raises the following questions:

- `Q1`: Why is the existing bottom up approach of modeling metatechnologies different from the approach used in the standard nonparametric production context?
- `Q2`: What are the production axioms describing properties of the metatechnology itself and to which extent are these axioms similar or distinctive from the ones characterizing the group technologies?
- `Q3`: How can a metatechnology be modeled using the MEP model selection criterion in the same approach established in the standard nonparametric production context?
- `Q4`: Assuming that a new axiomatic approach for modeling metatechnologies can be established (affirmatively answering question `Q2`), to which extent is it equivalent to the existing bottom up approach?

We get rid of any ambiguity regarding the modeling of metatechnologies by answering all questions `Q1`–`Q4`. To answer question `Q1`, recall from Section 1 that any metatechnology and any of its group technologies are called supra-technology and infra-technology, respectively. Then, all existing models in the meta nonparametric production context describe properties of the supra-technology by using infra-technology specific axioms, but not supra-technology specific axioms. Clearly, this approach is different from the approach described in Subsection 2.1.

To answer question Q2, we start with interpreting any metatechnology similar to the general interpretation of a production technology as in (1). To be explicit, we interpret metatechnology $\mathcal{T}^{\mathcal{G}}$ with its corresponding group technologies $\mathcal{T}^g$, $g \in \mathcal{G}$, as follows:

$$\mathcal{T}^{\mathcal{G}} = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \mathbf{x} \text{ can produce } \mathbf{y}\right\}. \tag{5}$$

Clearly, in contrast to the original interpretation (4) of metatechnology $\mathcal{T}^{\mathcal{G}}$, this new interpretation (5) does not make any prior link between the metatechnology itself and its group technologies. In particular, our proposed interpretation does not make the prior assumption that the metatechnology satisfies axiom GGT, but rather allows to infer this satisfaction from our new axiomatic definitions of metatechnologies (see Theorems 4.1 and 5.1).

To describe properties of the metatechnology $\mathcal{T}^{\mathcal{G}}$ based on its new interpretation (5), we introduce the following metatechnology specific axioms:

Axiom IGT (*Inclusion of Group Technologies*) Metatechnology $\mathcal{T}^{\mathcal{G}}$ includes all of its group technologies. That is, $\mathcal{T}^g \subseteq \mathcal{T}^{\mathcal{G}}$, for all $g \in \mathcal{G}$.

Axiom GIO (*Group-wise Inclusion of Observations*) For each $g \in \mathcal{G}$, the observations in subgroup g are included in group technology $\mathcal{T}^g$. That is, $\mathcal{N}_g \subset \mathcal{T}^g$, for all $g \in \mathcal{G}$.

Axiom WGSD (*Within-Group Strong Disposability*) For each $g \in \mathcal{G}$, we have $(\mathcal{T}^g + \mathcal{C}) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}^g$, where $\mathcal{C}$ is the free disposal cone considered in axiom SD.

Axiom Δ-WGRS (Δ-*Within-Group Returns to Scale*) For each $g \in \mathcal{G}$, we have $\delta \mathcal{T}^g \subseteq \mathcal{T}^g$ for all $\delta \in \mathcal{I}_\Delta$, where the set Δ is as considered in axiom Δ-RS.

Axiom WGC (*Within-Group Convexity*) For each $g \in \mathcal{G}$, group technology $\mathcal{T}^g$ is convex.

Although the trivial axiom IGT guarantees that the union of group technologies $\mathcal{T}^g$, $g \in \mathcal{G}$, is embedded into the metatechnology $\mathcal{T}^{\mathcal{G}}$, it does not incorporate the opposite of this embedding (i.e., $\mathcal{T}^{\mathcal{G}} \subseteq \bigcup_{g \in \mathcal{G}} \mathcal{T}^g$) in contrast to the axiom GGT introduced in Subsection 2.2. This means that axiom IGT is a weaker assumption than axiom GGT. As we demonstrate in the next sections, using the IGT axiom in our new modeling approach suffices for axiomatically modeling metatechnologies: the prior incorporation of the opposite embedding $\mathcal{T}^{\mathcal{G}} \subseteq \bigcup_{g \in \mathcal{G}} \mathcal{T}^g$ is not *mathematically* necessary.

The next axioms GIO, WGSD, Δ-WGRS and WGC are extensions to axioms IO, SD, Δ-RS and C, respectively. Specifically, if there is only one group technology (i.e., $G = 1$), then axioms GIO, WGSD, Δ-WGRS and WGC coincide with axioms IO, SD, Δ-RS and C, respectively. Therefore, to the best of our knowledge, the introduced metatechnology specific axioms are new.

As the metatechnology normally satisfies axiom IGT, axiom GIO is stronger than axiom IO. That is, if metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axioms IGT and GIO, then it also satisfies axiom IO, but the converse is in general not true. This is because the inclusion of observations in the metatechnology does not imply that the observations in each subgroup be included in their corresponding group technology. The above statement is highlighted as the following proposition.

Proposition 3.1. *Let metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfy axiom IGT. If metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axiom GIO, then it satisfies axiom IO.*

If metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axiom GGT, then axiom WGSD implies axiom SD. Otherwise, no general relation holds between these axioms.

We acknowledge that any real-life metatechnology represents a specific type of returns to scale. Furthermore, to the best of our knowledge, there is no empirical evidence that the returns to scale of group technologies differ.[10] Nonetheless, implementing axiom Δ-WGRS in this contribution, we exclude any circumstances where group technologies exhibit different types of returns to scale.

Bear in mind that axiom Δ-RS imposes an specific assumption regarding the scaling of all production units in metatechnology $\mathcal{T}^{\mathcal{G}}$. While axiom Δ-WGRS appears very similar to axiom Δ-RS, these axioms are generally not identical. Specifically, if metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axiom GGT, then the common returns to scale characteristic is transferred from the group technologies to the metatechnology and vice versa. Therefore, if metatechnology $\mathcal{T}^{\mathcal{G}}$ is traditionally interpreted as in (4), then the stated identity holds true. However, if our new interpretation (5) of metatechnology $\mathcal{T}^{\mathcal{G}}$ is admitted, then axiom Δ-WGRS assumes only a common returns to scale for each of its group technologies, but does not imply any statement regarding the returns to scale of metatechnology $\mathcal{T}^{\mathcal{G}}$. Indeed, because our interpretation does not restrict metatechnology $\mathcal{T}^{\mathcal{G}}$ to satisfy axiom GGT, it does not prohibit (mathematically) metatechnology $\mathcal{T}^{\mathcal{G}}$ from any potential inclusion of additional units which do not belong to any of its group technologies. Therefore, without satisfaction of axiom GGT, it cannot generally be inferred that a metatechnology exhibits the same type of returns to scale commonly exhibited by its group technologies. For illustration, consider the following example.

Example 3.1. Consider the six observations $A, \ldots, F$ shown in Fig. 1, where each uses a single input x to produce a single output y. Suppose that these observations are divided into two disjoint subgroups as $\mathcal{N}_1 = \{A, B, C\}$ and $\mathcal{N}_2 = \{D, E, F\}$. Further, consider the convex VRS technologies $\mathcal{T}^1$ and $\mathcal{T}^2$ as group technologies for each of the drawn metatechnologies $\hat{\mathcal{T}}^{\mathcal{G}}$, $\bar{\mathcal{T}}^{\mathcal{G}}$ and $\tilde{\mathcal{T}}^{\mathcal{G}}$.

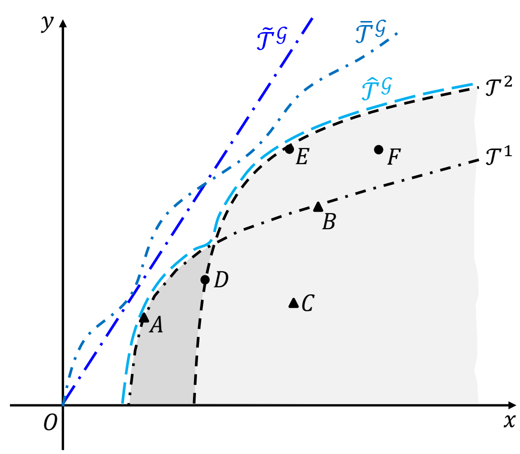


Figure 1: Returns to scale of metatechnology and its group technologies

From Fig. 1, it is clear that each of the metatechnologies $\hat{\mathcal{T}}^{\mathcal{G}}$, $\bar{\mathcal{T}}^{\mathcal{G}}$ and $\tilde{\mathcal{T}}^{\mathcal{G}}$ satisfies axioms IGT, GIO, WGSD, V-WGRS and WGC. Additionally, in contrast to metatechnologies $\bar{\mathcal{T}}^{\mathcal{G}}$ and $\tilde{\mathcal{T}}^{\mathcal{G}}$, metatechnology $\hat{\mathcal{T}}^{\mathcal{G}}$ satisfies axiom GGT. Therefore, as expected, metatechnology $\hat{\mathcal{T}}^{\mathcal{G}}$ has the same VRS characteristic of group technologies $\mathcal{T}^1$ and $\mathcal{T}^2$. However, metatechnology $\tilde{\mathcal{T}}^{\mathcal{G}}$ exhibits CRS.

[10] Note that it is theoretically possible to build a metatechnology with a specific returns to scale by taking the union of a finite number of group technologies with different returns to scale assumptions. For example, consider the two observations $A = (1, 1)$ and $B = (2, 2)$, where the first and second components are input x and output y, respectively. Hypothetically, suppose that the standard group technologies $\mathcal{T}^{\min}_{\text{ND-C}}$ and $\mathcal{T}^{\min}_{\text{NI-C}}$ are induced by units A and B, respectively. If the union of these group technologies is considered as a metatechnology induced by units A and B, then this metatechnology satisfies the CRS assumption.

Note that metatechnology $\bar{\mathcal{T}}^{\mathcal{G}}$ exhibits VRS, though it does not satisfy axiom GGT. This shows that the satisfaction of axiom GGT is sufficient, but not necessary, for the identity of the returns to scale characteristic of a metatechnology with that of its group technologies.

Remark that in the remainder, metatechnologies can maintain any of the returns to scale characteristics spelled out in axiom Δ-WGRS.

Proposition 3.2. *Let metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfy axiom GGT. Then, the following statements are true:*

(i) If metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axiom WGSD, then it satisfies axiom SD.

(ii) If metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axiom Δ-WGRS, then it satisfies axiom Δ-RS.

The proofs of Proposition 3.2 and the other statements are given in Appendix A.

It is worth noting that the group specific convexity axiom WGC is weaker than the metatechnology specific convexity axiom C.[11] Namely, a metatechnology with more than one convex group technology is in general nonconvex. For example, consider metatechnology $\hat{\mathcal{T}}^{\mathcal{G}}$ with its group technologies $\mathcal{T}^1$ and $\mathcal{T}^2$ in Example 3.1. While both of group technologies are convex, metatechnology $\hat{\mathcal{T}}^{\mathcal{G}}$ is nonconvex.

4 Within-Group Convex Metatechnology

4.1 Existing and Newly Proposed Models

The traditional approach of modeling metatechnologies can be considered as a bottom up (BU) approach because it starts with characterizing its group (or infra) technologies. In this section, we develop a new top down (TD) axiomatic approach that applies the metatechnology specific axioms introduced in Section 3 for modeling metatechnologies. Our proposed approach extends the approach described in Section 2.1 from the standard to the meta nonparametric production context. In particular, our approach reduces to the standard approach if there is only one group technology.

For each $g \in \mathcal{G}$, the notation $\mathcal{T}^g \sqsubseteq \mathcal{T}^{\mathcal{G}}$ shows that $\mathcal{T}^g$ is the gth group technology of metatechnology $\mathcal{T}^{\mathcal{G}}$. Additionally, $\mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}$ denotes the Δ-C group technology generated by the observations in subgroup $\mathcal{N}_g$. We denote the Δ-WGC metatechnology modeled by the BU and TD approaches by $\mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-WGC}}$ and $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$, respectively. For each $g \in \mathcal{G}$, we denote the gth group technology of these metatechnologies by $\mathcal{T}^{g\text{-BU}}_{\Delta\text{-WGC}}$ and $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$, respectively.

In the traditional approach of modeling metatechnologies, first, technologies $\mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}$, $g \in \mathcal{G}$, are considered to be approximations of the true group technologies. That is, the Δ-WGC group technologies are defined as

$$\mathcal{T}^{g\text{-BU}}_{\Delta\text{-WGC}} = \mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}, \quad g \in \mathcal{G}. \tag{6}$$

Then, based on interpretation (4), the Δ-WGC metatechnology $\mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-WGC}}$ is defined as the union of these group technologies:

$$\mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-WGC}} = \bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}. \tag{7}$$

[11] The concept of within-group convexity (WGC) is inspired by an informal discussion in Cooper et al. (2007, p. 231): "The convexity assumption holds within the same system but does not hold between the two systems."

Because the axioms used in the development of model (7) describe properties of the Δ-WGC group technologies, we regard the traditional approach as a BU approach. The special CRS and VRS cases of metatechnology $\mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-WGC}}$ are considered in Kerstens et al. (2019): see Proposition 5.5(b) and (a).

To develop our TD axiomatic model of the true metatechnology, we denote $\mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ the set of all metatechnologies $\mathcal{T}^{\mathcal{G}}$ that satisfy axioms IGT, GIO, WGSD, Δ-WGRS and WGC:

$$\mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \left\{ \mathcal{T}^{\mathcal{G}} \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \mathcal{T}^{\mathcal{G}} \text{ satisfies axioms IGT, GIO, WGSD, } \Delta\text{-WGRS and WGC} \right\}. \tag{8}$$

Then, we make the following definition based on our interpretation (5) of metatechnologies.

Definition 4.1. The Δ-WGC metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ and its corresponding group technologies are defined, respectively, as follows:

$$\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}} = \bigcap_{\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}} \mathcal{T}^{\mathcal{G}}, \tag{9a}$$

$$\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} = \bigcap_{\substack{\mathcal{T}^{g} \sqsubseteq \mathcal{T}^{\mathcal{G}},\\ \mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}}} \mathcal{T}^{g}, \quad g \in \mathcal{G}. \tag{9b}$$

Let us consider the special case that there is only one group technology, i.e., $G = 1$. Then, Definition 4.1 turns to the definition of the standard Δ-C technology, and question `Q3` is answered. By the next result, we develop an equivalent statement of metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ as the union of its group technologies $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-C}}$, $g \in \mathcal{G}$. This statement shows that metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ fulfills axiom GGT, without incorporating this axiom as a prior assumption.

Theorem 4.1. *The following statement is true:*

$$\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}} = \bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}. \tag{10}$$

The next result proves that metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ satisfies the MEP in terms of the axioms considered in (8).

Theorem 4.2. *Metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ satisfies the MEP in terms of axioms IGT, GIO, WGSD, Δ-WGRS and WGC.*

By Propositions 3.1 and 3.2, it follows from Theorem 4.2 that metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ inherits the properties SD and Δ-RS from its group technologies. Formally stated, metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ satisfies axioms IO, SD and Δ-RS.

We now answer question `Q4` by establishing the identity of the existing BU and our proposed TD models of the Δ-WGC metatechnology in the following theorem.

Theorem 4.3. *The following statements are true:*

(i) *For all* $g \in \mathcal{G}$, $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} = \mathcal{T}^{g\text{-BU}}_{\Delta\text{-WGC}} = \mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}$.

(ii) $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}} = \mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-WGC}} = \bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}$.

Thus, (i) each group technology $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$ defined by our proposed TD approach equals the corresponding group technology $\mathcal{T}^{g\text{-BU}}_{\Delta\text{-WGC}}$ defined by the conventional BU approach, and (ii) our proposed TD and the conventional BU models of the Δ-WGC metatechnology are equal. These statements are important because their development gets rid of any ambiguity arising from the use of group technology specific axioms in modeling the Δ-WGC metatechnologies. Based on Theorem 4.3, the superscripts 'TD' and 'BU' are not used in the remainder of this paper while denoting the Δ-WGC metatechnology and its group technologies.

By the next theorem, we establish two new results between the reference WGC metatechnologies that exhibit different types of returns to scale.

Theorem 4.4. *The following statements are true:*

(i) $\mathcal{T}^{\mathcal{G}}_{\text{C-WGC}} = \mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}} \cup \mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$.

(ii) $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}} \cap \mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$.

Theorem 4.4 is described as follows: (i) the C-WGC metatechnology coincides with the union of its corresponding NI- and ND-WGC metatechnologies, and (ii) the V-WGC metatechnology is included in the intersection of the NI- and ND-WGC metatechnologies. However, as illustrated below in Example 4.1, the opposite embedding is generally not true.

Note that Theorem 4.4 can be regarded as an extension to Proposition 1 in Briec et al. (2000). Namely, if there is one group technology, i.e., $G = 1$, then Theorem 4.4 reduces to Proposition 1 in Briec et al. (2000). We illustrate Theorem 4.4 by the following example.

Example 4.1. Consider the seven observations $A, \ldots, G$ defined in Table 1, and assume that $\mathcal{N}_1 = \{A, B, C, D\}$ and $\mathcal{N}_2 = \{E, F, G\}$.

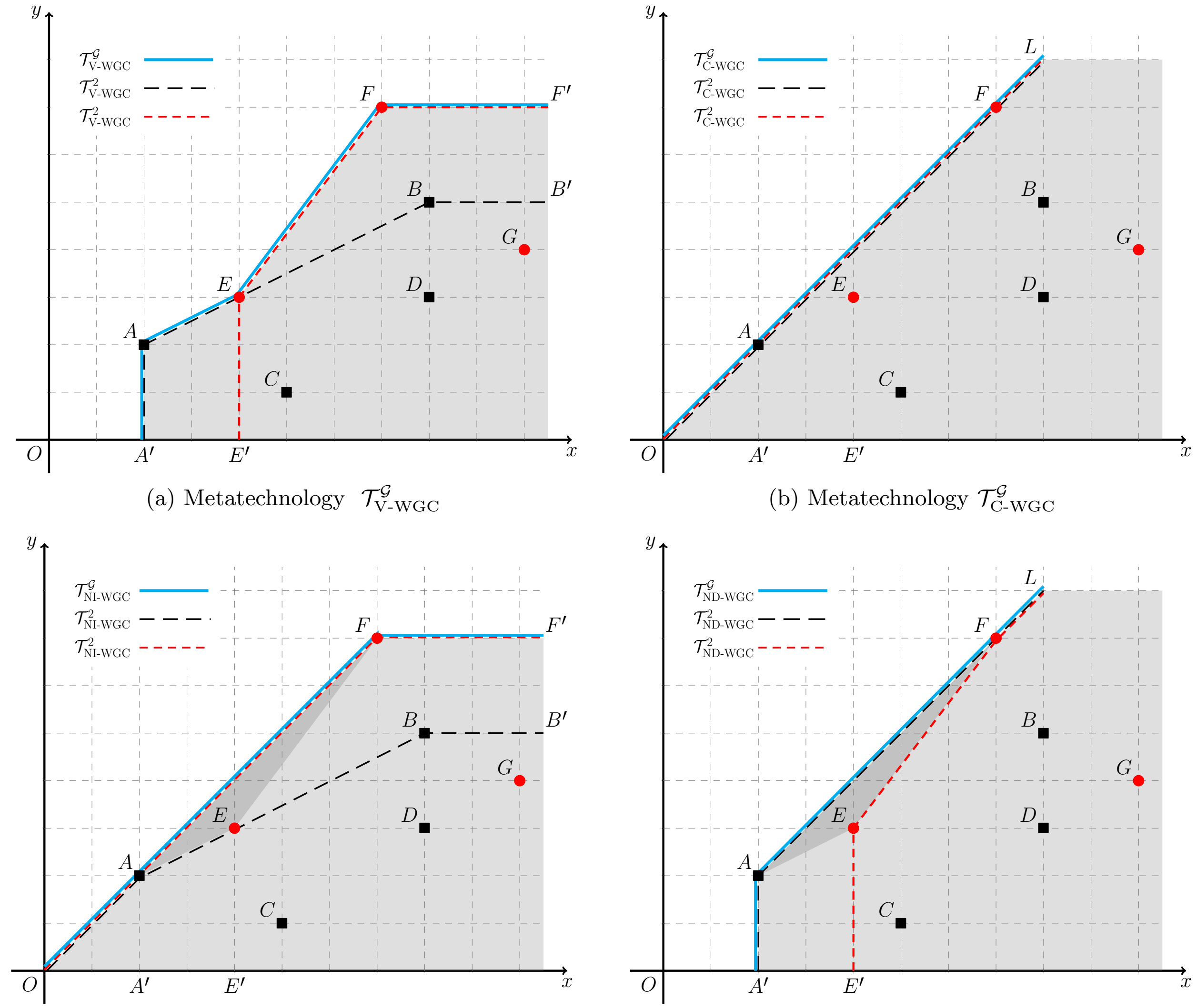
Table 1: The data set in Example 4.1

Unit	A	B	C	D	E	F	G
x	2	8	5	8	4	7	10
y	2	5	1	3	3	7	4

The shaded area below and right of the line $A'ABB'$ in Fig. 2a shows the V-C group technology $\mathcal{T}^{1}_{\text{V-C}}$ generated by the observations in $\mathcal{N}_1$. Additionally, the shaded area below and right of the line $E'EFF'$ shows the V-C group technology $\mathcal{T}^{2}_{\text{V-C}}$ generated by the observations in $\mathcal{N}_2$. The union of group technologies $\mathcal{T}^{1}_{\text{V-C}}$ and $\mathcal{T}^{2}_{\text{V-C}}$, which is the shaded area below and right of the line $A'AEFF'$ in Fig. 2a, shows the $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ metatechnology generated by all of the seven observations. Clearly, this metatechnology is nonconvex.

The CRS, NIRS and NDRS extensions of the generated VRS metatechnology are depicted in Fig. 2b, 2c and 2d, respectively. Note that group technologies of the CRS metatechnology are identical: the shaded area below the ray OL. The first and second group technologies of the NIRS reference metatechnology are the shaded areas below and right of the lines $OABB'$ and OFF', respectively. Further, the first and second group technologies of the NDRS reference metatechnology are the shaded areas below and right of the lines $A'AL$ and $E'EFL$, respectively.

It is observed from Fig. 2 that, in contrast to the nonconvex metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$, all three CRS, NIRS and NDRS reference metatechnologies are convex. As expected by statement (i) of Theorem 4.4,



(a) Metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$

(b) Metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{C-WGC}}$

(c) Metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}}$

(d) Metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$

Figure 2: The reference WGC metatechnologies in Example 4.1.

metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{C-WGC}}$ is equal to the union of metatechnologies $\mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}}$ and $\mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$. Furthermore, as expected by statement (ii) of Theorem 4.4, the intersection of metatechnologies $\mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}}$ and $\mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$ includes metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$. However, it is realized from comparing Fig. 2a with Fig. 2c and 2d that the opposite of this inclusion is not true. This is, because metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ does not include the relative interior of segment AF and the interior of triangle AEF that lie in both metatechnologies $\mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}}$ and $\mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$ and, therefore, in their intersection.

Remark 4.1. *Let us define the intersection of metatechnologies $\mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}}$ and $\mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$ as a metatechnology that its group technologies are $\mathcal{T}^{g}_{\text{NI-WGC}} \cap \mathcal{T}^{g}_{\text{ND-WGC}}$, $g \in \mathcal{G}$. By Proposition 1 in Briec et al. (2000), it follows that $\mathcal{T}^{g}_{\text{NI-WGC}} \cap \mathcal{T}^{g}_{\text{ND-WGC}} = \mathcal{T}^{g\text{-min}}_{\text{V-WGC}}$ for all $g \in \mathcal{G}$. Then, it can be verified that $\mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}} \cap \mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$ is a VRS metatechnology that satisfies all axioms used in the definition of metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$. However, in contrast to metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ (see Theorem 4.2), the stated intersection does not generally meet the MEP in terms of axioms used in the definition of metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$. This is due to the fact that the opposite of the embedding given in statement (ii) of Theorem 4.4 is not generally true. For illustration, refer to Example 3.1.*

4.2 Algebraic Statements

In this section we propose two algebraic statements of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. To the best of our knowledge, the first statement has been widely used in the literature, but without any formal presentation (except in the VRS case). In contrast, the second novel statement is original and has not been presented elsewhere. By the next result, we propose the first statement of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$.

Theorem 4.5. *Metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ is equivalently stated as follows:*

$$
\begin{aligned}
\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid & \sum_{g\in\mathcal{G}} \sum_{j\in\mathcal{J}_g} \lambda^g_j \mathbf{x}_j \leq \mathbf{x}, \\
& \sum_{g\in\mathcal{G}} \sum_{j\in\mathcal{J}_g} \lambda^g_j \mathbf{y}_j \geq \mathbf{y}, \\
& \mathbf{1}^\top \boldsymbol{\lambda}^g = \delta\gamma_g,\ \ g \in \mathcal{G}, \\
& \delta \in \mathcal{I}_\Delta, \\
& \mathbf{1}^\top \boldsymbol{\gamma} = 1, \\
& \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{\gamma} \in \{0,1\}^G\Big\}.
\end{aligned} \tag{11}
$$

The set on the right hand-side of (11) incorporates a vector of binary variables γ_g, $g \in \mathcal{G}$, such that variable γ_g corresponds to subgroup g. The sum of these binary variables is restricted to be equal to one. Therefore, for any unit in this metatechnology set, there exists a $g' \in \mathcal{G}$ such that the binary variable $\gamma_{g'}$ takes the value of one, while all remaining binary variables are zero. Because this unit may belong simultaneously to different group technologies, g' denotes one of these group technologies.

The statement of Theorem 4.5 for $\Delta = \mathrm{V}$ has earlier appeared in Huang et al. (2013) without any formal proof. In this case, all characterization conditions of the set on the right hand-side of equality (11) are linear. This is not, however, true in the cases of NIRS, NDRS and CRS, because the variables γ_g, $g \in \mathcal{G}$, are multiplied with the scaling variable δ. Clearly, this nonlinearity is present even if unit $(\mathbf{x},\mathbf{y})$ is given.

The following corollary of Theorem 4.5 identifies a special case in which two types of generally nonconvex technologies – technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ and metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ – are equal to one another. In this case, each subgroup of observations includes exactly one observation and the corresponding group technology is equal to the strong disposal hull of this observation.

Corollary 4.1. *Let $\mathcal{N}_j = \{(\mathbf{x}_j, \mathbf{y}_j)\}$ for all $j \in \mathcal{J}$. Then, the following statement is true:*

$$
\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \mathcal{T}^{\min}_{\Delta\text{-NC}}. \tag{12}
$$

The next novel result proposes the second statement of the Δ-WGC metatechnology. Note that the conditions of this statement are linear for any given unit $(\mathbf{x},\mathbf{y}) \in \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$.

Theorem 4.6. *Metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ is equivalently stated as follows:*

$$\begin{aligned}\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid & \sum_{j\in\mathcal{J}_g} \lambda^g_j \mathbf{x}_j \leq \gamma_g \mathbf{x}, \quad g \in \mathcal{G}, \\ & \sum_{j\in\mathcal{J}_g} \lambda^g_j \mathbf{y}_j \geq \gamma_g \mathbf{y}, \quad g \in \mathcal{G}, \\ & \mathbf{1}^\top \boldsymbol{\lambda}^g - \gamma_g \in \mathcal{D}_\Delta, \quad g \in \mathcal{G}, \\ & \mathbf{1}^\top \boldsymbol{\gamma} = 1, \\ & \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{\gamma} \in \{0,1\}^G \Big\},\end{aligned} \tag{13}$$

where $\mathcal{D}_{\text{NI}} = -\mathbb{R}_+$, $\mathcal{D}_{\text{V}} = \{0\}$, $\mathcal{D}_{\text{ND}} = \mathbb{R}_+$ and $\mathcal{D}_{\text{C}} = \mathbb{R}$.

The idea of developing our algebraic statement (13) of the WGC metatechnology is to incorporate the input and output specific conditions for all group technologies, and introduce the binary variables γ_g, $g \in \mathcal{G}$, corresponding to the WGC group technologies. In addition, we impose the condition $\mathbf{1}^\top \boldsymbol{\gamma} = 1$. Then, for any unit in the set on the right-hand side of (13), only one binary variable, e.g., $\gamma_g = \gamma_{g'}$, can take the value of one, while the remaining variables, e.g., γ_g, $g \neq g'$, must take the common value of zero. Therefore, the input and output specific conditions are not affected only for $g = g'$, whereas these conditions disappear for all $g \neq g'$.

We remark that statements (11) and (13) of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ are valuable from the modeling point of view. Indeed, the availability of these statement allows to formulate the problem of estimating any performance measure relative to metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ as a single optimization program. However, as argued in the next subsections, using the second statement (13) for making such formulations is more efficient than the first one (11) from the computational point of view. Furthermore, the new statement (13) helps reveal alternative implicit ideas behind some existing results.

4.3 Measurement of Directional Technical Inefficiency

Consider the statement (11) of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. Then, measuring technical (in)efficiency in this metatechnology by any of Farrell input or output measures (see, e.g., Färe et al. (1985)) and the directional distance function (Chambers et al. (1998)) leads to solving a mixed 0-1 program. This program is linear only in the case of VRS and cannot be equivalently relaxed to its continuous form. Thus, there may be some concern regarding its computational efficiency in large samples. Of course, there should not be such concern because there are alternative linear programming based approaches for computing the efficiency score. Specifically, the technical efficiency can be obtained by solving G linear programs (see Huang et al. (2013) and Kerstens et al. (2019)). In this approach, the efficiency of the unit evaluated with respect to the Δ-WGC metatechnology is given by the minimum of its within-group efficiencies. The idea of this approach is that optimizing any function over the metatechnology is equivalent to finding the best value obtained from optimizing this function over the corresponding group technologies.

In this section, we use our proposed statement (13) of the Δ-WGC metatechnology for the measurement of technical (in)efficiency. Our results are developed based on the directional distance function, and reduce simply to the cases of Farrell input and output efficiency. The same remark applies to the efficiency defined by Farrell proportional distance function of Briec (1997).

Let $(\mathbf{x}_o, \mathbf{y}_o)$ with $\mathbf{x}_o \neq \mathbf{0}$ and $\mathbf{y}_o \neq \mathbf{0}$ denote the production unit whose efficiency is being evaluated with respect to metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. We assume that it is one of the observations or, more generally, any production unit in this metatechnology. Then, the directional distance function based on metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ is defined as

$$\begin{aligned} \tau_o^{\Delta}(\mathbf{d}) = \max \quad & \tau \\ \text{s.t.} \quad & (\mathbf{x}_o - \tau \mathbf{d}_x, \mathbf{y}_o + \tau \mathbf{d}_y) \in \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}, \\ & \tau \text{ sign free}, \end{aligned} \tag{14}$$

where $\mathbf{d} = (\mathbf{d}_x, \mathbf{d}_y) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \backslash \{(\mathbf{0}, \mathbf{0})\}$ is a user-specified direction vector.

Clearly, the optimal value $\tau_o^{\Delta}(\mathbf{d})$ is always non-negative, and can be interpreted in general as the technical inefficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$. Let this optimal value be attained at τ^*, and define the corresponding projection unit as $(\mathbf{x}^*_o, \mathbf{y}^*_o) = (\mathbf{x}_o - \tau^* \mathbf{d}_x, \mathbf{y}_o + \tau^* \mathbf{d}_y)$. Then it is straightforward to verify that this unit is efficient in the direction of $\mathbf{d}$. Therefore, unit $(\mathbf{x}_o, \mathbf{y}_o)$ is technical efficient in the direction of $\mathbf{d}$ if and only if $\tau_o^{\Delta}(\mathbf{d}) = 0$, and is inefficient otherwise.

Let unit $(\mathbf{x}_o, \mathbf{y}_o)$ be in group technology $\mathcal{T}^{g_o}_{\Delta\text{-WGC}}$, where $g_o \in \mathcal{G}$. Then, its corresponding projection unit $(\mathbf{x}^*_o, \mathbf{y}^*_o)$ is restricted by the optimization program (14) to be situated only in metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$, and not necessarily in its group technology $\mathcal{T}^{g_o}_{\Delta\text{-WGC}}$. This implies that unit $(\mathbf{x}_o, \mathbf{y}_o)$ is improved with respect to all observations, and not only the observations in group technology $\mathcal{T}^{g_o}_{\Delta\text{-WGC}}$.

Based on the statement (13) of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$, program (14) can be restated in the following expanded form:

$$\tau_o^{\Delta}(\mathbf{d}) = \max \quad \tau \tag{15a}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}_g} \lambda^g_j \mathbf{x}_j \leq \gamma_g \mathbf{x}_o - \gamma_g \, \tau \, \mathbf{d}_x, \quad g \in \mathcal{G}, \tag{15b}$$

$$\sum_{j \in \mathcal{J}_g} \lambda^g_j \mathbf{y}_j \geq \gamma_g \mathbf{y}_o + \gamma_g \, \tau \, \mathbf{d}_y, \quad g \in \mathcal{G}, \tag{15c}$$

$$\mathbf{1}^\top \boldsymbol{\lambda}^g - \gamma_g \in \mathcal{D}_\Delta, \;\; g \in \mathcal{G}, \tag{15d}$$

$$\mathbf{1}^\top \boldsymbol{\gamma} = 1, \tag{15e}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \; \boldsymbol{\gamma} \in \{0, 1\}^G, \; \tau \text{ sign free}. \tag{15f}$$

Though program (15) is a mixed 0-1 nonlinear optimization program, we transform it into an equivalent linear program. First, we show that program (15) is equivalent to its continuous form (see Lemma A.2 in Appendix A). After that, we implement the variable substitutions $\tau_g = \gamma_g \, \tau$, $g \in \mathcal{G}$, to linearize the resulting continuous program as follows:

$$\hat{\tau}_o^{\Delta}(\mathbf{d}) = \max \quad \sum_{g\in\mathcal{G}} \tau_g \tag{16a}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{J}_g} \lambda_j^g \mathbf{x}_j \leq \gamma_g \mathbf{x}_o - \tau_g \mathbf{d}_x, \quad g \in \mathcal{G}, \tag{16b}$$

$$\sum_{j\in\mathcal{J}_g} \lambda_j^g \mathbf{y}_j \geq \gamma_g \mathbf{y}_o + \tau_g \mathbf{d}_y, \quad g \in \mathcal{G}, \tag{16c}$$

$$\mathbf{1}^\top \boldsymbol{\lambda}^g - \gamma_g \in \mathcal{D}_\Delta, \quad g \in \mathcal{G}, \tag{16d}$$

$$\mathbf{1}^\top \boldsymbol{\gamma} = 1, \tag{16e}$$

$$\boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{\gamma} \geq \mathbf{0},\ \boldsymbol{\tau} \text{ sign free.} \tag{16f}$$

Theorem 4.7. *The optimal values of programs* (15) *and* (16) *are equal:* $\tau_o^{\Delta}(\mathbf{d}) = \hat{\tau}_o^{\Delta}(\mathbf{d})$.

There are special cases where program (14), and equivalently programs (15) and (16), can provide the efficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$. Specifically, if the direction vector is chosen as $\mathbf{d} = (\mathbf{x}_o, \mathbf{0})$, $\mathbf{d} = (\mathbf{0}, \mathbf{y}_o)$, and $\mathbf{d} = (\mathbf{x}_o, \mathbf{y}_o)$, then the values $1 - \tau_o^{\Delta}(\mathbf{d})$, $\frac{1}{1+\tau_o^{\Delta}(\mathbf{d})}$ and $1 - \tau_o^{\Delta}(\mathbf{d})$ are interpreted, respectively, as Farrell input, output and proportional efficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$.

Note that program (16) unifies the linear program (16) developed by Afsharian and Podinovski (2018) as well as its suggested variants into a single optimization program. This may raise the question as to why their results are reproduced here. The answer is that this recovery clarifies the fact that an alternative unknown idea behind the approach of Afsharian and Podinovski (2018) is to develop the directional distance function with respect to our proposed statement (13) of the Δ-WGC metatechnology. Note that the approach of Afsharian and Podinovski (2018) for the single-stage estimation of the directional technical inefficiency with respect to metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ is to first develop a mixed 0-1 nonlinear program without any explicit statement of the Δ-WGC metatechnology, and then to linearize that program.

Note also that, as described in Subsection 5.2, other special cases of program (16) are the linear programs developed by Agrell and Tind (2001) and Leleu (2006) for the measurement of Farrell input efficiency in the standard Δ-NC technology.

4.4 Measurement of Directional Economic Inefficiency

In this section, we introduce a new measure of directional economic efficiency that encompasses the conventional cost, revenue and profit efficiency measures as special cases. In contrast to the conventional measures, our measure allows to evaluate the economic efficiency with respect to any subvector of inputs and outputs. We discuss how to estimate this measure based on the statement (13) of metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$.

We denote $\mathbf{w} \in \mathbb{R}^m_{++}$ and $\mathbf{q} \in \mathbb{R}^s_{++}$ the price vectors of inputs and outputs, respectively. Let $\mathbf{d} = (\mathbf{d}_x, \mathbf{d}_y) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \backslash \{(\mathbf{0}, \mathbf{0})\}$ be a user-specified direction vector, and let $\mathcal{I}^+$ and $\mathcal{O}^+$ denote the index sets of strictly positive components of the direction subvectors $\mathbf{d}_x$ and $\mathbf{d}_y$:

$$\mathcal{I}^+ = \{i \in \mathcal{I} \mid d_{xi} > 0\}, \qquad \mathcal{O}^+ = \{r \in \mathcal{O} \mid d_{yr} > 0\}. \tag{17}$$

Based on (17), we define the *Actual Economic Value* (AEV) of any unit $(\mathbf{x}_j, \mathbf{y}_j)$ in the direction of $\mathbf{d}$ as $\sum_{r\in\mathcal{O}^+} q_r y_r - \sum_{i\in\mathcal{I}^+} w_i x_i$. For each $j \in \mathcal{J}$, we denote E_j the AEV of observation $(\mathbf{x}_j, \mathbf{y}_j)$. We also denote E_o and E_o^* the AEVs of unit $(\mathbf{x}_o, \mathbf{y}_o)$ and, respectively, its projection unit $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ obtained from program (14).

Let $\bar{\mathbf{d}} = \left(\bar{\mathbf{d}}_x, \bar{\mathbf{d}}_y\right) = (\mathbf{w} \otimes \mathbf{d}_x, \mathbf{q} \otimes \mathbf{d}_y)$. Then, $\mathbf{1}^\top \bar{\mathbf{d}} = \sum_{i\in\mathcal{I}^+} w_i d_{xi} + \sum_{r\in\mathcal{O}^+} q_r d_{yr} > 0$, and the directional technical inefficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$ in metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ can be stated as follows:

$$\tau_o^{\text{V}}(\mathbf{d}) = \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}}(E_o^* - E_o). \tag{18}$$

We define the *Directional Economic Inefficiency* of unit $(\mathbf{x}_o, \mathbf{y}_o)$ with respect to metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ as the optimal value $\varepsilon_o(\mathbf{d})$ of the following new optimization program:

$$\begin{aligned} \varepsilon_o(\mathbf{d}) = \max \quad & \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}}\Big(\sum_{i\in\mathcal{I}^+} \bar{d}_{xi}\beta_{xi} + \sum_{r\in\mathcal{O}^+} \bar{d}_{yr}\beta_{yr}\Big) \\ \text{s.t.} \quad & \left(\mathbf{x}_o - \boldsymbol{\beta}_x \otimes \mathbf{d}_x, \mathbf{y}_o + \boldsymbol{\beta}_y \otimes \mathbf{d}_y\right) \in \mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}, \\ & \boldsymbol{\beta} \text{ sign free.} \end{aligned} \tag{19}$$

Clearly, the optimal value of program (19) is always non-negative. Let this optimal value be attained at $\boldsymbol{\beta}^{**}$, and define the corresponding projection unit as $(\mathbf{x}_o^{**}, \mathbf{y}_o^{**}) = \left(\mathbf{x}_o - \boldsymbol{\beta}_x^{**} \otimes \mathbf{d}_x, \mathbf{y}_o + \boldsymbol{\beta}_y^{**} \otimes \mathbf{d}_y\right)$. Furthermore, denote E_o^{**} the AEV of this projection unit, which is in fact the *maximum* economic value achievable by unit $(\mathbf{x}_o, \mathbf{y}_o)$. Then, $\varepsilon_o(\mathbf{d})$ can be stated as the normalized difference between the maximum and actual economic values of unit $(\mathbf{x}_o, \mathbf{y}_o)$:

$$\varepsilon_o(\mathbf{d}) = \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}}(E_o^{**} - E_o). \tag{20}$$

Therefore, equality (20) confirms our interpretation of $\varepsilon_o(\mathbf{d})$ as the economic inefficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$ in the direction of $\mathbf{d}$. Based on this interpretation, we call unit $(\mathbf{x}_o, \mathbf{y}_o)$ *economic efficient* in the direction of $\mathbf{d}$ if and only if $\varepsilon_o(\mathbf{d}) = 0$, and economic inefficient otherwise.

Let us define the *directional allocative inefficiency* $\alpha_o(\mathbf{d})$ of unit $(\mathbf{x}_o, \mathbf{y}_o)$ in the direction of $\mathbf{d}$ as the normalized difference between the AEVs of the projection units obtained from programs (14) and (19):

$$\alpha_o(\mathbf{d}) = \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}}(E_o^{**} - E_o^*). \tag{21}$$

It is clear that $\alpha_o(\mathbf{d}) \geq 0$. By the three equalities (18), (20) and (21), the directional economic inefficiency is *additively* decomposed into the technical and allocative components as follows:

$$\varepsilon_o(\mathbf{d}) = \tau_o^{\text{V}}(\mathbf{d}) + \alpha_o(\mathbf{d}). \tag{22}$$

Note that when $\mathcal{I}^+ = \mathcal{I}$ and $\mathcal{O}^+ = \varnothing$ (resp., $\mathcal{I}^+ = \varnothing$ and $\mathcal{O}^+ = \mathcal{O}$), then our measure turns to the traditional cost (resp., revenue) inefficiency measure. Similarly, if $\mathcal{I}^+ = \mathcal{I}$ and $\mathcal{O}^+ = \mathcal{O}$, then our measure is transformed to the conventional profit inefficiency measure.

Based on statement (13) of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$, program (19) is restated in the following new expanded form:

$$\varepsilon_o(\mathbf{d}) = \max \ \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}} \Big(\sum_{i \in \mathcal{I}^+} \bar{d}_{xi} \beta_{xi} + \sum_{r \in \mathcal{O}^+} \bar{d}_{yr} \beta_{yr} \Big) \tag{23a}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}_g} \lambda_j^g \mathbf{x}_j \leq \gamma_g \mathbf{x}_o - \gamma_g \, \boldsymbol{\beta}_x \otimes \mathbf{d}_x, \quad g \in \mathcal{G}, \tag{23b}$$

$$\sum_{j \in \mathcal{J}_g} \lambda_j^g \mathbf{y}_j \geq \gamma_g \mathbf{y}_o + \gamma_g \, \boldsymbol{\beta}_y \otimes \mathbf{d}_y, \quad g \in \mathcal{G}, \tag{23c}$$

$$\mathbf{1}^\top \boldsymbol{\lambda}^g = \gamma_g, \quad g \in \mathcal{G}, \tag{23d}$$

$$\mathbf{1}^\top \boldsymbol{\gamma} = 1, \tag{23e}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \ \boldsymbol{\gamma} \in \{0,1\}^G, \ \boldsymbol{\beta} \text{ sign free.} \tag{23f}$$

The next result proves that the mixed 0-1 nonlinear optimization program (23) can be transformed equivalently into the following new linear program:

$$\hat{\varepsilon}_o(\mathbf{d}) = \max \ \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}} \Big(\sum_{g \in \mathcal{G}} \sum_{j \in \mathcal{J}_g} \lambda_j^g E_j - E_o \Big) \tag{24a}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}_g} \lambda_j^g x_{ij} \leq \gamma_g x_{io}, \quad i \in \mathcal{I} \backslash \mathcal{I}^+, \ g \in \mathcal{G}, \tag{24b}$$

$$\sum_{j \in \mathcal{J}_g} \lambda_j^g y_{rj} \geq \gamma_g y_{ro}, \quad r \in \mathcal{O} \backslash \mathcal{O}^+, \ g \in \mathcal{G}, \tag{24c}$$

$$\mathbf{1}^\top \boldsymbol{\lambda}^g = \gamma_g, \quad g \in \mathcal{G}, \tag{24d}$$

$$\mathbf{1}^\top \boldsymbol{\gamma} = 1, \tag{24e}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \ \boldsymbol{\gamma} \geq \mathbf{0}. \tag{24f}$$

Theorem 4.8. *The optimal values of programs* (23) *and* (24) *are equal:* $\varepsilon_o(\mathbf{d}) = \hat{\varepsilon}_o(\mathbf{d})$.

Note that there are three approaches for the measurement of directional economic inefficiency in the WGC metatechnology. First, we can solve the new mixed 0-1 nonlinear optimization program (23). Second, we can follow the traditional multi-stage approach: optimizing the objective function (23a) over each of the WGC group technologies, and then taking their maximum. Similar to the existing multi-stage approaches suggested for the measurement of technical efficiency in the literature, this approach leads again to solving G linear programs. Third, our approach is to solve the single-stage linear program (24): the computational efficiency gains are obvious.

The following corollary of Theorem 4.8 considers the special case that all inputs and outputs are incorporated into the measurement of directional economic inefficiency, and provides a very simple approach for its computation. Note that, in this case, each E_j becomes the same *observed profit* π_j of observation $(\mathbf{x}_j, \mathbf{y}_j)$.

Corollary 4.2. *Let* $\mathcal{I}^+ = \mathcal{I}$ *and* $\mathcal{O}^+ = \mathcal{O}$. *Then, the following statement is true:*

$$\varepsilon_o(\mathbf{d}) = \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}} \Big(\max_{g \in \mathcal{G}} \max_{j \in \mathcal{J}_g} \pi_j - \pi_o \Big). \tag{25}$$

Equality (25) implies that the maximum profit in metatechnology $\mathcal{T}_{\text{V-WGC}}^{\mathcal{G}}$ occurs at some of the observations and its value can be computed as $\max_{g\in\mathcal{G}} \max_{j\in\mathcal{J}_g} \pi_j$. It is worth noting that this result reduces to the equality (2.2) in Färe and Zelenyuk (2020) in the special case that there is only one group technology, i.e., $G = 1$.

Suppose that the observed profit π_o is known and all units face the same prices for inputs and outputs. Then, Corollary 4.2 shows that evaluating the profit efficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$ can be made without any information on prices and even on input–output vectors of observations. Indeed, it suffices to have only the observed profits of observations, and then determine their maximum value.

It is worth noting that our directional economic inefficiency measure is general in the sense that it allows to evaluate the economic inefficiency based on any non-empty subvector of inputs and outputs depending on the choice of direction vector. This is worthwhile from the economic point of view. Indeed, in economics it is common to differentiate in the short-run (SR) between fixed and variable inputs/outputs depending on whether inputs/outputs are exogenous to managerial control or are under complete control of management. In case all inputs and outputs are under managerial control, then we talk about long-run (LR) directional economic inefficiency.

Let $\mathbf{d}$ be a direction vector such that its positive components correspond to all selected (fixed and variable) inputs and outputs with respect to which the economic inefficiency is being evaluated. Then, the resulting directional economic, technical and allocative inefficiency measures are called LR, and are denoted by $\varepsilon_o^{\text{LR}}(\mathbf{d})$, $\tau_o^{\text{V-LR}}(\mathbf{d})$ and $\alpha_o^{\text{LR}}(\mathbf{d})$, respectively. Let $\mathbf{d}'$ denote the direction vector resulting from $\mathbf{d}$ by setting zero the values of components that correspond to the fixed inputs and outputs. Then, the resulting directional economic, technical and allocative inefficiency measures are called SR, and are denoted by $\varepsilon_o^{\text{SR}}(\mathbf{d}')$, $\tau_o^{\text{V-SR}}(\mathbf{d}')$ and $\alpha_o^{\text{SR}}(\mathbf{d}')$, respectively. Based on these definitions and notations, we can obtain the following statement between the LR and SR directional technical efficiency measures:

$$\tau_o^{\text{V-LR}}(\mathbf{d}) \geq \tau_o^{\text{V-SR}}(\mathbf{d}'). \tag{26}$$

However, no general relation can be established between the LR and SR directional economic inefficiency and, therefore, between the LR and SR directional allocative inefficiency.

5 Metatechnology Without Axiom WGC

5.1 Existing and Newly Proposed Models

For each $g \in \mathcal{G}$, let $\mathcal{T}_{\Delta\text{-NC}}^{g\text{-min}}$ denote the Δ-NC group technology generated by the observations in subgroup $\mathcal{N}_g$. We denote the Δ-NC metatechnology modeled by the BU and TD approaches by $\mathcal{T}_{\Delta\text{-NC}}^{\mathcal{G}\text{-BU}}$ and $\mathcal{T}_{\Delta\text{-NC}}^{\mathcal{G}\text{-TD}}$, respectively. For each $g \in \mathcal{G}$, we denote the gth group technology of these metatechnologies by $\mathcal{T}_{\Delta\text{-NC}}^{g\text{-BU}}$ and $\mathcal{T}_{\Delta\text{-NC}}^{g\text{-TD}}$, respectively.

By the conventional BU approach of modeling metatechnologies, the Δ-NC group technologies are defined as

$$\mathcal{T}_{\Delta\text{-NC}}^{g\text{-BU}} = \mathcal{T}_{\Delta\text{-NC}}^{g\text{-min}}, \quad g \in \mathcal{G}. \tag{27}$$

Then, based on interpretation (4), the Δ-NC metatechnology $\mathcal{T}_{\Delta\text{-NC}}^{\mathcal{G}\text{-BU}}$ is defined as the union of these group technologies:

$$\mathcal{T}_{\Delta\text{-NC}}^{\mathcal{G}\text{-BU}} = \bigcup_{g\in\mathcal{G}} \mathcal{T}_{\Delta\text{-NC}}^{g\text{-min}}. \tag{28}$$

To give an alternative statement of metatechnology $\mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-NC}}$, let us define the Δ-strong disposal hull of any unit $(\mathbf{x}, \mathbf{y})$ as $\{(\delta\tilde{\mathbf{x}}, \delta\tilde{\mathbf{y}}) \mid \delta \in \mathcal{I}_\Delta,\ \tilde{\mathbf{x}} \geq \mathbf{x},\ \mathbf{0} \leq \tilde{\mathbf{y}} \leq \mathbf{y}\}$, where $\mathcal{I}_\Delta$ is as considered in axiom Δ-RS. Then the set on the right-hand side of (28) can be stated as the union of the Δ-strong disposal hull of all observations (see Briec et al. (2004, p. 164)), which coincides with the nonconvex technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ induced by all observations. Taking into account this equality, the following statement results from (28):

$$\mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-NC}} = \mathcal{T}^{\min}_{\Delta\text{-NC}}. \tag{29}$$

Equality (29) implies that the Δ-NC metatechnology, in addition to its group technologies, is generally nonconvex. Therefore, if any real-world metatechnology makes no convexity assumption at all, then there is no need for incorporating its meta-structure into its modeling. Indeed, the nonparametric specification of such a metatechnology is given by the nonconvex Δ-NC technology induced by all observations. It is worth recalling that the special CRS and VRS versions of the Δ-NC metatechnology have been considered by Kerstens et al. (2019). In particular, equality (29) duplicates their Proposition 5.5(e) and (d).

In this section, we develop a TD axiomatic approach to recover model (28) from the metatechnology specific axioms introduced in Section 3 (excluding axiom WGC). Our proposed approach is based on our interpretation (5), but not the traditional interpretation (4), of metatechnologies. As a nonconvex counterpart of $\mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ defined in (8), we denote $\mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ the set of all metatechnologies that satisfy axioms IGT, GIO, WGSD and Δ-WGRS. Formally, assume that

$$\mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}} = \left\{\mathcal{T}^{\mathcal{G}} \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \mathcal{T}^{\mathcal{G}} \text{ satisfies axioms IGT, GIO, WGSD and } \Delta\text{-WGRS}\right\}. \tag{30}$$

By using (30), we make the following axiomatic definition of the Δ-NC metatechnology.

Definition 5.1. The Δ-NC metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$ and its corresponding group technologies are defined, respectively, as follows:

$$\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}} = \bigcap_{\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}} \mathcal{T}^{\mathcal{G}}, \tag{31a}$$

$$\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}} = \bigcap_{\substack{\mathcal{T}^{g} \sqsubseteq \mathcal{T}^{\mathcal{G}},\\ \mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}}} \mathcal{T}^{g}, \quad g \in \mathcal{G}. \tag{31b}$$

The equivalent statement of metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$ in the next theorem shows that this metatechnology satisfies axiom GGT.

Theorem 5.1. *The following statement is true:*

$$\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}} = \bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}}. \tag{32}$$

The next result shows that metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$ satisfies the MEP in terms of its defining axioms.

Theorem 5.2. *Metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$ satisfies the MEP in terms of axioms IGT, GIO, WGSD and Δ-WGRS.*

By Propositions 3.1 and 3.2, it follows from Theorem 5.2 that the Δ-NC metatechnology satisfies axioms IO, SD and Δ-RS.

By the next result, we demonstrate the equivalence between the existing BU and our proposed TD approaches of modeling the Δ-NC metatechnology.

Theorem 5.3. *The following statements are true:*

(i) For all $g \in \mathcal{G}$, $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}} = \mathcal{T}^{g\text{-BU}}_{\Delta\text{-NC}} = \mathcal{T}^{g\text{-min}}_{\Delta\text{-NC}}$.

(ii) $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}} = \mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-NC}} = \mathcal{T}^{\text{min}}_{\Delta\text{-NC}}$.

Thus, (i) each group technology $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}}$ defined by our TD approach equals the corresponding group technology $\mathcal{T}^{g\text{-BU}}_{\Delta\text{-NC}}$ defined by the conventional BU approach, (ii) our proposed TD and the conventional BU models of the Δ-NC metatechnology are equal to the Δ-NC technology defined over all observations. Based on Theorem 5.3, we do not use the superscripts 'TD' and 'BU' in the remainder of this paper while denoting the Δ-NC metatechnology and its group technologies.

By Corollary 4.1 and Theorem 5.3, we demonstrate the equality of Δ-WGC and Δ-NC metatechnologies in a special case.

Corollary 5.1. *Let* $\mathcal{N}_j = \{(\mathbf{x}_j, \mathbf{y}_j)\}$ *for all* $j \in \mathcal{J}$. *Then, the following statement is true:*

$$\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}.$$

This result is used in the next section for developing a statement of metatechnology $\mathcal{T}^{\text{min}}_{\Delta\text{-NC}}$.

Based on statement (ii) of Theorem 5.3, two statements are established in the next result by using the results suggested by Kerstens and Vanden Eeckaut (1999) and Briec et al. (2004).

Theorem 5.4. *The following statements are true:*

(i) $\mathcal{T}^{\mathcal{G}}_{\text{C-NC}} = \mathcal{T}^{\mathcal{G}}_{\text{NI-NC}} \cup \mathcal{T}^{\mathcal{G}}_{\text{ND-NC}}$.

(ii) $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\text{NI-NC}} \cap \mathcal{T}^{\mathcal{G}}_{\text{ND-NC}}$.

Thus, (i) the C-NC metatechnology is the union of the reference NI- and ND-NC metatechnologies, and (ii) the V-NC metatechnology is included in the intersection of the NI-NC and ND-NC metatechnologies, but the converse is not generally true. Note that a remark similar to Remark 4.1 applies to $\mathcal{T}^{\mathcal{G}}_{\text{NI-NC}} \cap \mathcal{T}^{\mathcal{G}}_{\text{ND-NC}}$: although this intersection is a VRS metatechnology that satisfies all axioms used in the definition of metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$, it is not necessarily the smallest one.

Since metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ satisfies axioms IGT, GIO, WGSD and Δ-WGRS, the following embedding follows from Theorem 5.2:

$$\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}. \tag{33}$$

Note that metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ essentially convexifies the group technologies of the nonconvex metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$. Formally, taking into account Theorems 4.1 and 4.3, the following equality is obtained by applying (3) to group technologies of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$:

$$\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \bigcup_{g \in \mathcal{G}} \text{conv}\big(\mathcal{T}^{g\text{-min}}_{\Delta\text{-NC}}\big). \tag{34}$$

Example 5.1. Let $A, \ldots, G$ be the same observations considered in Example 4.1, and let $\mathcal{N}_1 = \{A, B, C, D\}$ and $\mathcal{N}_2 = \{E, F, G\}$. Then, the shaded area below and right of the line $A'AUBB'$ in Fig. 3 shows the V-NC group technology $\mathcal{T}^1_{\text{V-NC}}$ generated by the observations in $\mathcal{N}_1$. Additionally, the shaded area below and right of the line $E'EVFF'$ shows the V-NC group technology $\mathcal{T}^2_{\text{V-NC}}$ generated by the observations in $\mathcal{N}_2$. The union of group technologies $\mathcal{T}^1_{\text{V-NC}}$ and $\mathcal{T}^2_{\text{V-NC}}$ is the shaded area below and right of the line $A'AWEVFF'$, and shows the $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$ metatechnology generated by all of the seven observations. Note that the CRS, NIRS and NDRS extensions of metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$ are equal to the metatechnologies depicted in Fig. 2b, 2c and 2d, respectively.

Clearly, metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$ is the same nonconvex technology $\mathcal{T}^{\min}_{\text{V-NC}}$ generated by the observations. From comparing Fig. 2a and Fig. 3, it is observed that metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ convexifies only the group technologies of metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$, but not the whole of this metatechnology.

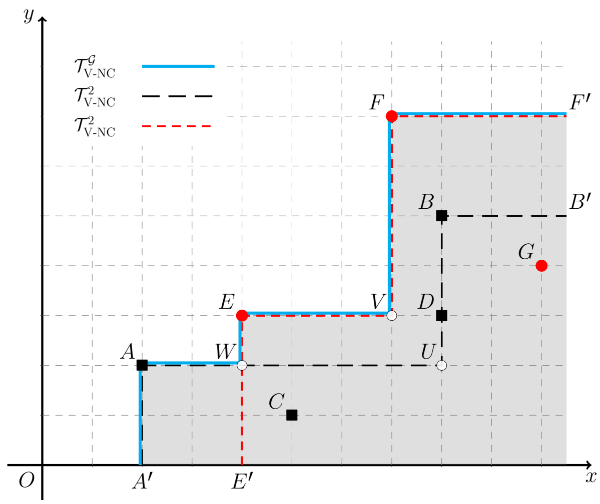


Figure 3: Metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$ in Example 5.1

5.2 Algebraic Statements and Measurement of Directional Technical Inefficiency

By Theorem 5.3, metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ equals the standard technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$. Taking this into account, one equivalent statement of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ is given in this section using the results established for technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ in the literature. Another additional alternative statement is proposed by exploiting Corollary 5.1. The measurement of technical efficiency based on each of these statement is discussed. Then the section is concluded by suggesting a computationally efficient approach for measuring the technical efficiency.

From the identity of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ and technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ as established in Theorem 5.3, it follows that an equivalent statement of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ is the nonlinear one appeared in the right hand-sides of (2). Based on this statement, the Farrell input efficiency with respect to metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ can be measured by solving the nonlinear mixed 0-1 programs of Kerstens and Vanden Eeckaut (1999). It is worth noting that the computational efficiency of this approach is improved by Podinovski (2004) who develops the linear variants of these programs using a "big-M" linearization technique. By the point stated in Podinovski (2004), we note that the Farrell output efficiency with respect to metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ can be measured by making obvious changes in his proposed programs. We also note that a similar comment applies to the measurement of Farrell proportional efficiency. The directional distance function based (in)efficiency with respect to metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$ can be measured by the linear mixed 0-1 program developed by Cherchye et al. (2001). Extending this program to the non-VRS cases results in the nonlinear mixed 0-1 programs developed in Kerstens and Van de Woestyne (2018).

By Corollary 5.1, the Δ-WGC and Δ-NC metatechnologies are identical if $\mathcal{J}_j = \{j\}$ for all $j \in \mathcal{J}$. Incorporating this fact into the statement (13) of the Δ-WGC metatechnology, an alternative statements of the Δ-NC metatechnology is given in the next result. Note that the set $\mathcal{D}_\Delta$ is as defined in Theorem 4.6.

Theorem 5.5. *The following statement is true:*

$$\begin{aligned}\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}} = \big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid\ & \lambda_j \mathbf{x}_j \leq \gamma_j \mathbf{x},\ j \in \mathcal{J}, \\ & \lambda_j \mathbf{y}_j \geq \gamma_j \mathbf{y},\ j \in \mathcal{J}, \\ & \lambda_j - \gamma_j \in \mathcal{D}_\Delta,\ j \in \mathcal{J}, \\ & \mathbf{1}^\top \boldsymbol{\gamma} = 1, \\ & \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{\gamma} \in \{0,1\}^n\big\}.\end{aligned} \tag{35}$$

Under the assumption of VRS, Agrell and Tind (2001) show how the Farrell input efficiency with respect to technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ can be measured by solving the first stage of a two-stage linear program. Leleu (2006) presents the single-stage form of this program as linear program (P1) in his work. Additionally, he develops linear program (P2) in his work to extend this result to the cases of NIRS, NDRS and CRS. Note that formulating the Farrell input efficiency measure based on statement (35) of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ leads to the linear programs of Agrell and Tind (2001) and Leleu (2006). In particular, it follows from Corollary 5.1 and Theorem 5.5 that their linear programs are special cases of program (16). Therefore, an alternative unknown idea behind developing the linear programs (P1) and (P2) of Leleu (2006) is to develop the Farrell input efficiency measure with respect to the statement (35) of the Δ-NC technology.

Having developed the two alternative statements (2) and (35) of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$, we consider the question as to which statement is more advantageous than the others. We answer to this question from different perspectives. From the modeling point of view, any of the two given statements can be used for formulating single-stage optimization programs by which the technical efficiency is measured. If the optimization program resulting from any statement is a linear program, then the duality of linear programming allows for enhancing the economic interpretation of the nonconvex metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ in terms of shadow prices. For example, the dual interpretation suggested by Agrell and Tind (2001) and Leleu (2006) applies to metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ because their linear programs can be obtained from statement (35).

By the above argument, any algebraic statement of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ such that the optimization program resulting from which is a linear program appears better than any other statements based on which mixed 0-1 programs are developed. This may suggest using the linear programming based approaches for the measurement of directional technical inefficiency in metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$. This suggestion is based on the known fact that the computational burden of solving linear programs are normally less than those of mixed 0-1 programs. However, it is empirically shown by Briec and Kerstens (2006) and Kerstens and Van de Woestyne (2014) that the computational efficiency of the enumeration algorithms used in Deprins et al. (1984), Lovell (1995), Tulkens (1993), Briec et al. (2004) and Briec and Kerstens (2006), Cherchye et al. (2001) and Kerstens and Van de Woestyne (2018) for the measurement of technical efficiency in technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$ is more than solving linear programs. Therefore, from the equality of metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ with technology $\mathcal{T}^{\min}_{\Delta\text{-NC}}$, the use of the enumeration algorithms of Cherchye et al. (2001) and Kerstens and Van de Woestyne (2018) for the measurement of directional technical inefficiency in metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}}$ is computationally recommended.

5.3 Measurement of Directional Economic Inefficiency

Let $\mathbf{d}$ be any direction vector as considered in Subsections 4.3 and 4.4. Then, we define the directional economic inefficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$ in metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$ is defined as the optimal value of the following program:

$$\begin{aligned} e_o(\mathbf{d}) = \max \quad & \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}} \Big(\sum_{i \in \mathcal{I}^+} \bar{d}_{xi} \beta_{xi} + \sum_{r \in \mathcal{O}^+} \bar{d}_{yr} \beta_{yr} \Big) \\ \text{s.t.} \quad & \big(\mathbf{x}_o - \boldsymbol{\beta}_x \otimes \mathbf{d}_x, \mathbf{y}_o + \boldsymbol{\beta}_y \otimes \mathbf{d}_y \big) \in \mathcal{T}^{\mathcal{G}}_{\text{V-NC}}, \\ & \boldsymbol{\beta} \text{ sign free.} \end{aligned} \tag{36}$$

Similar to the decomposition of $\varepsilon_o(\mathbf{d})$ as in (22), the directional economic inefficiency $e_o(\mathbf{d})$ is decomposed into the technical and allocative components as

$$e_o(\mathbf{d}) = t^{\text{V}}_o(\mathbf{d}) + a_o(\mathbf{d}), \tag{37}$$

where $t^{\text{V}}_o(\mathbf{d})$ is the directional technical inefficiency of unit $(\mathbf{x}_o, \mathbf{y}_o)$ in metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$, and $a_o(\mathbf{d})$ is the deviation between the directional economic and technical inefficiencies of this unit.

As mentioned in the previous subsection, $t^{\text{V}}_o(\mathbf{d})$ can be estimated by using the enumeration algorithm suggested by Cherchye et al. (2001). By the next result, we show that the value of $e_o(\mathbf{d})$ can be obtained by solving the following linear program:

$$\hat{e}_o(\mathbf{d}) = \max \quad \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}} \Big(\sum_{j \in \mathcal{J}} \gamma_j E_j - E_o \Big) \tag{38a}$$

$$\text{s.t.} \quad \gamma_j x_{ij} \le \gamma_j x_{io}, \quad i \in \mathcal{I} \backslash \mathcal{I}^+, \ j \in \mathcal{J}, \tag{38b}$$

$$\gamma_j y_{rj} \ge \gamma_j y_{ro}, \quad r \in \mathcal{O} \backslash \mathcal{O}^+, \ j \in \mathcal{J}, \tag{38c}$$

$$\mathbf{1}^\top \boldsymbol{\gamma} = 1, \tag{38d}$$

$$\boldsymbol{\gamma} \ge \mathbf{0}. \tag{38e}$$

Theorem 5.6. *The optimal values of programs* (36) *and* (38) *are equal:* $e_o(\mathbf{d}) = \hat{e}_o(\mathbf{d})$.

Just as in the case of Theorem 4.8, the single-stage linear program (38) has obvious computational efficiency gains. The following corollary of Theorem 5.6 takes one more step and further simplifies the computation of $e_o(\mathbf{d})$ in the special case that all inputs and outputs are taken into account.

Corollary 5.2. *Let* $\mathcal{I}^+ = \mathcal{I}$ *and* $\mathcal{O}^+ = \mathcal{O}$. *Then, the following statement is true:*

$$e_o(\mathbf{d}) = \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}} \Big(\max_{g \in \mathcal{G}} \max_{j \in \mathcal{J}_g} \pi_j - \pi_o \Big). \tag{39}$$

The interpretation of Corollary 5.2 is that the maximum profit in metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ occurs at some of the observations and its value can be computed as $\max_{g \in \mathcal{G}} \max_{j \in \mathcal{J}_g} \pi_j$. This result can be regarded as an extension of the equality (2.3) in Färe and Zelenyuk (2020).

A useful consequence of Corollaries 4.2 and 5.2 is that the maximum profits estimated in metatechnologies $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ and $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$ are identical for any direction vector $\mathbf{d} > \mathbf{0}$:

$$\varepsilon_o(\mathbf{d}) = e_o\left(\mathbf{d}\right). \tag{40}$$

Therefore, the absence or presence of the convexity of group technologies $\mathcal{T}^{g}_{\text{V-WGC}}$ (stated formally as axiom WGC) does not affect the measurement of directional profit inefficiency.

Note that the concepts of LR and SR directional economic, technical and allocative inefficiencies can also be defined with respect to the NC metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$, similar to those introduced at the end of Subsection 4.4. To save space, the superscripts "LR" and "SR" are used to denote the resulting directional economic, technical and allocative inefficiency measures.

6 Erroneous Convexification of the Δ-WGC Metatechnology

In this section, we introduce a special version of convexity that allows for convex combinations of observations in an erroneous metatechnology which are not present in any of the group technologies themselves, but rather are situated in between the group technologies. We show that adding this axiom into the definition of the WGC metatechnology leads to an erroneous metatechnology which convexifies the axiomatically correct WGC metatechnology. Kerstens et al. (2019) label this potentially wrong estimation strategy as a convexification strategy, and empirically illustrate that such a convexification strategy leads to a substantially biased metafrontier. In the remainder, we call this a *pseudo-metatechnology*.

Consider the Δ-C technology $\mathcal{T}^{\min}_{\Delta\text{-C}}$ induced by all observations as defined in Section 2.1. To establish a relationship between this convex technology and the within-group convex metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$, we introduce the following special version of the convexity axiom at the level of the pseudo-metatechnology:

Axiom BGC (*Between-Group Convexity*) For any $g, g' \in \mathcal{G}$ such that $g \neq g'$, we have $\lambda\mathcal{T}^{g} + (1-\lambda)\mathcal{T}^{g'} \subseteq \mathcal{T}^{\mathcal{G}}$ for all $\lambda \in (0,1)$.

For the sake of completeness, we define a pseudo-metatechnology adopting an erroneous convexification strategy by assuming axiom BGC. Specifically, let $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ denote the Δ-C pseudo-metatechnology defined as the intersection of all pseudo-metatechnologies that satisfy axioms IGT, GIO, WGSD, Δ-WGRS, WGC and BGC. Pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ satisfies both axioms WGC and BGC and, therefore, is convex. Furthermore, it satisfies axioms IGT, GIO, WGSD, Δ-WGRS and WGC. Because metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ satisfies the MEP in terms of these axioms (see Theorem 4.2), the following embedding is true:

$$\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}. \tag{41}$$

Therefore, pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ does not satisfy the MEP in terms of axioms IGT, GIO, WGSD, Δ-WGRS and WGC.

The next result establishes the equivalence between the Δ-C pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ and the Δ-C technology induced by all observations.

Theorem 6.1. *The following statement is true:*

$$\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}} = \mathcal{T}^{\min}_{\Delta\text{-C}}. \tag{42}$$

Theorem 6.1 shows that if a pseudo-metatechnology is counter factually assumed to be convex, then there is no need for incorporating its meta-structure into its modeling. In this particular case, the Δ-C technology induced by all observations is the model of the pseudo-metatechnology.

Based on the proof of Theorem 6.1, let us introduce technologies $\mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}$, $g \in \mathcal{G}$, as group technologies of pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$. It follows that $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ satisfies axioms IO, SD, Δ-RS and C. Additionally, the following result is obtained by Proposition 1 in Briec et al. (2000):

Theorem 6.2. *The following statements are true:*

(i) $\mathcal{T}^{\mathcal{G}}_{\text{C-C}} = \mathcal{T}^{\mathcal{G}}_{\text{NI-C}} \cup \mathcal{T}^{\mathcal{G}}_{\text{ND-C}}$.

(ii) $\mathcal{T}^{\mathcal{G}}_{\text{V-C}} = \mathcal{T}^{\mathcal{G}}_{\text{NI-C}} \cap \mathcal{T}^{\mathcal{G}}_{\text{ND-C}}$.

The next result shows that pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ convexifies metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ by adding axiom BGC to the axioms used in the definition of the latter metatechnology.

Theorem 6.3. *The following statement is true:*

$$\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}} = \text{conv}\left(\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}\right). \tag{43}$$

By the following corollary, we summarize the relationship between the authentic and pseudo-metatechnologies that are axiomatically developed in this contribution.

Corollary 6.1. *The following statement is true:*

$$\mathcal{T}^{\min}_{\Delta\text{-NC}} = \mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}} = \mathcal{T}^{\min}_{\Delta\text{-C}}. \tag{44}$$

The next example illustrates the convex structure of pseudo-metatechnology and the difference between the two convexities implemented by axioms WGC and BGC.

Example 6.1. Let $A, \ldots, G$ be the same observations considered in Example 4.1, and let $\mathcal{N}_1 = \{A, B, C, D\}$ and $\mathcal{N}_2 = \{E, F, G\}$. Then, Fig. 4 shows the pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-C}}$ generated by all observations $A, \ldots, G$. As expected from Theorem 6.1, this metatechnology is the same convex technology $\mathcal{T}^{\min}_{\text{V-C}}$ generated by these observations.

By comparing Fig. 2a and Fig. 4, it is seen that triangles ABU and EFV lie in both metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ and pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-C}}$. This is due to making the assumption that these metatechnologies satisfy axiom WGC. However, the interior of triangle AEF and the relative interior of segment AF are not included in metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$, but are in pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-C}}$. This is due to incorporating the additional axiom BGC that allows for taking the convex combinations of units in the two different subgroups, in addition to those of the units in the same subgroup.

7 Numerical Illustration

To illustrate the measurement of directional economic inefficiency and its subsequent decomposition into directional technical and directional allocative components as defined in (22) and (37), we employ a

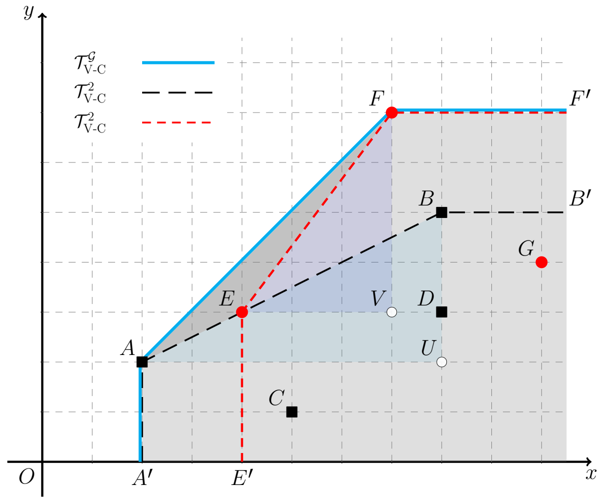


Figure 4: Pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\text{V-C}}$

numerical example conceived by Afsharian and Podinovski (2018, Table 3). To save space, we suppress the word directional in the remainder of this section. This data set consists of 32 Decision Making Units (DMUs) distributed across four group technologies: groups 1 to 4 consisting of 9, 12, 11, and 8 DMUs, respectively. Note that 6 of these DMUs are associated with multiple groups. Each DMU has three strictly positive inputs and two strictly positive outputs. We assume that $\mathbf{w} = (2, 1, 2)$ and $\mathbf{q} = (1, 2)$ are the price vectors of inputs and outputs, respectively. Furthermore, in line with Sections 4.4 and 5.3, we make the VRS assumption.

In this example, we consider the decomposition of both SR and LR economic inefficiencies in the WGC and NC metatechnologies. For the LR case, we assume that $\mathcal{I}^+ = \mathcal{I} = \{1, 2, 3\}$ and $\mathcal{O}^+ = \mathcal{O} = \{1, 2\}$. This yields the conventional profit inefficiency measure. For the SR case, we partition inputs and outputs into fixed and variable components: the second input and the second output are fixed, while the remaining inputs and outputs are variable. Thus, $\mathcal{I}^+ = \{1, 3\}$ and $\mathcal{O}^+ = \{1\}$. Since the input and output vectors are strictly positive for all DMUs, we can choose the direction vectors in the LR and SR cases as $\mathbf{d} = (x_{1o}, x_{2o}, x_{3o}, y_{1o}, y_{2o})$ and $\mathbf{d'} = (x_{1o}, 0, x_{3o}, y_{1o}, 0)$, respectively.

Table 2 displays the decomposition results of both LR and SR economic inefficiencies for the WGC and NC metatechnologies $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}}$ and $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}}$. Table 2 comprises thirteen columns. The first column identifies the DMU. Following the decomposition (22), the next six columns present the LR and SR economic, technical and allocative inefficiency scores measured relative to the WGC metatechnology. Following the decomposition (37), the last six columns show similarly structured results with respect to the NC metatechnology.

The first 32 rows of Table 2 represent all of the above described results for individual DMUs. The final five rows present descriptive statistics: arithmetic average, standard deviation, minimum value, maximum value, and the number of efficient observations (i.e., DMUs with a zero inefficiency score).

First, we start by analyzing the results based on the WGC metatechnology. We can draw the following conclusions from the decomposition starting with economic inefficiency and then its components. *(i)* In both the LR and SR cases, most DMUs exhibit economic inefficiency. Specifically, only DMU 3 is economic efficient in the LR, while seven DMUs are economic efficient in the SR. As expected, the relation between the LR and SR economic inefficiency cannot be signed: some DMUs experience higher economic inefficiency in the LR compared to the SR, while for others it is the other way around.

Table 2: Decomposing SR and LR directional economic inefficiency following (22) and (37)

	WGC metatechnology						NC metatechnology					
	LR			SR			LR			SR		
DMUs	$\varepsilon_o^{\mathrm{LR}}(\mathbf{d})$	$\tau_o^{\mathrm{V\text{-}LR}}(\mathbf{d})$	$\alpha_o^{\mathrm{LR}}(\mathbf{d})$	$\varepsilon_o^{\mathrm{SR}}(\mathbf{d})$	$\tau_o^{\mathrm{V\text{-}SR}}(\mathbf{d})$	$\alpha_o^{\mathrm{SR}}(\mathbf{d})$	$e_o^{\mathrm{LR}}(\mathbf{d}')$	$t_o^{\mathrm{V\text{-}LR}}(\mathbf{d}')$	$a_o^{\mathrm{LR}}(\mathbf{d}')$	$e_o^{\mathrm{SR}}(\mathbf{d}')$	$t_o^{\mathrm{V\text{-}SR}}(\mathbf{d}')$	$a_o^{\mathrm{SR}}(\mathbf{d}')$
1	0.087	0.000	0.087	0.000	0.000	0.000	0.087	0.000	0.087	0.000	0.000	0.000
2	0.161	0.127	0.034	0.200	0.150	0.050	0.161	0.115	0.046	0.200	0.143	0.057
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.374	0.113	0.260	0.562	0.353	0.209	0.374	0.071	0.302	0.562	0.333	0.229
5	0.147	0.000	0.147	0.000	0.000	0.000	0.147	0.000	0.147	0.000	0.000	0.000
6	0.174	0.093	0.081	0.266	0.175	0.091	0.174	0.052	0.122	0.266	0.138	0.128
7	0.235	0.000	0.235	0.014	0.000	0.014	0.235	0.000	0.235	0.000	0.000	0.000
8	0.799	0.733	0.066	0.831	0.795	0.036	0.799	0.720	0.079	0.831	0.765	0.065
9	0.565	0.383	0.182	0.693	0.470	0.223	0.565	0.293	0.272	0.693	0.443	0.250
10	0.364	0.126	0.238	0.256	0.188	0.068	0.364	0.101	0.263	0.240	0.168	0.072
11	0.169	0.000	0.169	0.000	0.000	0.000	0.169	0.000	0.169	0.000	0.000	0.000
12	0.310	0.067	0.243	0.180	0.095	0.086	0.310	0.050	0.260	0.155	0.070	0.085
13	0.195	0.000	0.195	0.004	0.000	0.004	0.195	0.000	0.195	0.000	0.000	0.000
14	0.213	0.000	0.213	0.008	0.000	0.008	0.213	0.000	0.213	0.000	0.000	0.000
15	0.336	0.079	0.257	0.239	0.121	0.118	0.336	0.051	0.285	0.225	0.097	0.129
16	0.181	0.000	0.181	0.002	0.000	0.002	0.181	0.000	0.181	0.000	0.000	0.000
17	0.261	0.076	0.185	0.245	0.124	0.121	0.261	0.044	0.217	0.173	0.114	0.059
18	0.691	0.529	0.163	0.785	0.605	0.181	0.691	0.484	0.207	0.785	0.554	0.231
19	0.142	0.035	0.107	0.045	0.036	0.010	0.142	0.035	0.108	0.037	0.035	0.002
20	0.231	0.059	0.172	0.200	0.091	0.110	0.231	0.056	0.175	0.186	0.064	0.122
21	0.224	0.024	0.200	0.122	0.024	0.098	0.224	0.005	0.219	0.094	0.005	0.089
22	0.107	0.000	0.107	0.000	0.000	0.000	0.107	0.000	0.107	0.000	0.000	0.000
23	0.343	0.121	0.222	0.325	0.279	0.046	0.343	0.087	0.256	0.285	0.261	0.024
24	0.284	0.134	0.150	0.230	0.157	0.073	0.284	0.129	0.156	0.196	0.146	0.050
25	0.246	0.078	0.168	0.199	0.164	0.035	0.246	0.054	0.191	0.174	0.164	0.010
26	0.154	0.000	0.154	0.000	0.000	0.000	0.154	0.000	0.154	0.000	0.000	0.000
27	0.184	0.000	0.184	0.000	0.000	0.000	0.184	0.000	0.184	0.000	0.000	0.000
28	0.220	0.053	0.168	0.112	0.081	0.031	0.220	0.000	0.220	0.112	0.048	0.065
29	0.292	0.109	0.183	0.272	0.122	0.150	0.292	0.058	0.234	0.204	0.073	0.130
30	0.297	0.132	0.165	0.283	0.137	0.146	0.297	0.091	0.206	0.175	0.100	0.075
31	0.294	0.123	0.171	0.280	0.163	0.117	0.294	0.076	0.219	0.190	0.135	0.055
32	0.185	0.026	0.159	0.099	0.069	0.030	0.185	0.000	0.185	0.000	0.000	0.000
Average	0.264	0.101	0.164	0.202	0.137	0.064	0.264	0.080	0.184	0.181	0.120	0.060
Stand.Dev.	0.163	0.161	0.062	0.230	0.186	0.066	0.163	0.152	0.070	0.230	0.178	0.073
Minimum	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Maximum	0.799	0.733	0.260	0.831	0.795	0.223	0.799	0.720	0.302	0.831	0.765	0.250
# Eff. DMUs	1	11	1	7	11	7	1	13	1	12	12	12

Note that economic efficient observations are simultaneously technical and allocative efficient. *(ii)* In both LR and SR, eleven DMUs are technical efficient. However, despite this equal amount of technical efficient DMUs in both cases, some DMUs experience higher technical inefficiency in the SR compared to the LR as expected. *(iii)* In the LR, only DMU 3 is allocative efficient. In the SR, seven DMUs are allocative efficient.

Second, turning to the analysis of the decomposition results based on the NC metatechnology, we can infer the following conclusions. *(i)* In both the LR and SR cases, most DMUs exhibit economic inefficiency. Specifically, only DMU 3 is economic efficient in the LR, while in the SR twelve DMUs achieve economic efficiency. *(ii)* In both the LR and SR cases, most DMUs show technical inefficiency. However, there are thirteen technical efficient DMUs in the LR. In the SR, this number of technical efficient DMUs decreases to twelve. Despite the about equal number of technical efficient DMUs in both LR and SR, as expected some DMUs undergo higher technical inefficiency in the SR compared to the LR. *(iii)* In the LR, only DMU 3 is allocative efficient. In the SR, twelve DMUs are allocative efficient.

Comparing the results of WGC and NC metatechnologies in Table 2, we selectively highlight the following conclusions. *(i)* For the WGC metatechnology, the number of economic efficient DMUs increases from one in the LR to seven in the SR. The NC metatechnology shows an even greater improvement: the number of economic efficient DMUs increases from one in the LR to twelve in the SR. When considering technical inefficiency, the WGC metatechnology maintains the same set of thirteen DMUs as technical efficient in both LR and SR. By contrast, the NC metatechnology experiences a slight decrease in the number of technical efficient DMUs from thirteen in the LR to twelve in the SR. *(ii)* An important observation is that the economic inefficiency under the NC and WGC metatechnologies is identical in the LR (see equality (40)), but it differs for the SR case. In the LR, both NC and WGC metatechnologies identify one economic efficient DMU, and the average economic inefficiency is thus the same for both metatechnologies (0.264). Given that the economic inefficiency in the LR is identical for NC and WGC metatechnologies and given that the technical inefficiency component is different, inevitably the allocative inefficiency component differs.

(iii) In the SR, there is a difference in the economic inefficiency results between the two metatechnologies. The NC metatechnology identifies five more economic efficient DMUs (twelve DMUs) compared to the WGC metatechnology (seven DMUs). In additional, the average economic inefficiency is slightly lower for the NC (0.181) compared to the WGC metatechnology (0.202). Thus, in the SR the NC metatechnology identifies more economic efficient DMUs and yields lower economic inefficiency compared to the WGC metatechnology. This confirms in the metafrontier context the conjecture in Färe and Zelenyuk (2020) that SR economic inefficiency can be different between convex and nonconvex technologies.

8 Conclusions

In the original metatechnology approach initiated by O'Donnell et al. (2008), one follows implicitly a bottom up axiomatic approach by defining axioms on the group technologies and little or no attention is paid to the resulting axiomatic framework at the metatechnology level. This bottom up axiomatic approach leads to minimum extrapolation results at the level of the group technologies.

In this paper we have developed an alternative top down axiomatic approach by defining axioms directly on the metatechnology. We have shown that without satisfaction of axiom GGT one cannot in general infer that a metatechnology exhibits the same type of returns to scale commonly exhibited by its group technologies. Our top down axiomatic approach leads to some new metatechnology specific minimum extrapolation results for both the WGC and the NC metatechnologies. Our algebraic statements of these two metatechnologies are compatible with any of the traditional returns to scale characteristics. Furthermore, our modeling distinguishes various types of nonconvexities: one due to the absence of within-group convexity, and another one due to the absence of between-group convexity.

We find several relations between the WGC and NC metatechnologies and the pseudo-metatechnology. First, we make two comparisons: one between the NC metatechnology and the standard NC firm technology, and another between the pseudo-metatechnology and the standard convex firm technology. *(i)* The NC metatechnology is identical to the standard nonconvex firm technology. *(ii)* The pseudo-metatechnology is identical to the standard convex firm technology. Second, we establish the following relation between these three metatechnologies: the NC metatechnology is embedded in the WGC metatechnology, which itself is embedded in the erroneously convexified pseudo-metatechnology. Third, we show that the WGC metatechnology convexifies the NC metatechnology at the group-level, and that the convex pseudo-metatechnology convexifies the WGC metatechnology at both the group-level and the meta-level.

We position our novel statement of the WGC metatechnology within the framework of measuring directional economic inefficiency and its additive decomposition into technical and allocative components. We thereby distinguish between long-run and short-run analysis. Furthermore, we develop linear programs for measuring both the directional technical and economic inefficiencies. Within the NC metatechnology, we also suggest the use of well-known enumeration algorithms for the measurement of directional technical inefficiency, and we establish a linear program for measuring directional economic inefficiency. Extending two simple formulations developed by Färe and Zelenyuk (2020) from the standard firm technology to the metatechnology context, we prove that the directional profit inefficiency under the WGC and NC metatechnologies is identical in the long-run, but not so in short-run.

For reasons of space, we have been unable to explore the definition of metatechnology gap ratios or their difference-based equivalent in the framework of directional economic inefficiency. Given the vastness of the existing metafrontier literature, we think this development is rather straightforward and we leave this to the reader or for future work.

To the best of our knowledge, the determination of returns to scale at the level of a metatechnology remains relatively unexplored. It is important to stress that since the metatechnology is by definition nonconvex, the characterization of returns to scale for individual observations is somewhat different from the traditional convex case: this has been theoretically argued by Podinovski (2004) and empirically illustrated by Cesaroni et al. (2017) in the standard context (see also the recent contribution of Mostafaee and Soleimani-Damaneh (2020) for further methodological refinements). This is certainly a fruitful avenue for further research.

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Appendices: Electronic Companion (Online Supplement)

A Proofs

Proof of Proposition 3.2. *Part (i)* Let metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfy axioms GGT and WGSD. To prove that $\mathcal{T}^{\mathcal{G}}$ satisfies axiom SD, let $(\mathbf{x},\mathbf{y}) \in \mathcal{T}^{\mathcal{G}}$. Then, by axiom GGT, there exists a $g' \in \mathcal{G}$ such that $(\mathbf{x},\mathbf{y}) \in \mathcal{T}^{g'}$. By axiom WGSD, $((\mathbf{x},\mathbf{y}) + \mathcal{C}) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}^{g'}$, where $\mathcal{C}$ is the free disposal cone considered in axiom SD. Again, by axiom GGT, it follows that $((\mathbf{x},\mathbf{y}) + \mathcal{C}) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}^{\mathcal{G}}$. Therefore, $\left(\mathcal{T}^{\mathcal{G}} + \mathcal{C}\right) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}^{\mathcal{G}}$, and metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axiom SD.

Part (ii) Let metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfy axioms GGT and Δ-WGRS. To prove that $\mathcal{T}^{\mathcal{G}}$ satisfies axiom Δ-RS, let $(\mathbf{x},\mathbf{y}) \in \mathcal{T}^{\mathcal{G}}$. Then, by axiom GGT, there exists a $g' \in \mathcal{G}$ such that $(\mathbf{x},\mathbf{y}) \in \mathcal{T}^{g'}$. By axiom Δ-WGRS, we have $\delta(\mathbf{x},\mathbf{y}) \in \mathcal{T}^{g'}$ for all $\delta \in \mathcal{I}_\Delta$. Again, by axiom GGT, it follows that $\delta(\mathbf{x},\mathbf{y}) \in \mathcal{T}^{\mathcal{G}}$ for all $\delta \in \mathcal{I}_\Delta$. Therefore, metatechnology $\mathcal{T}^{\mathcal{G}}$ satisfies axiom SD. □

Lemma A.1. *For each $g \in \mathcal{G}$, the following statements are true:*

(i) $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$.

(ii) $\mathcal{N}_g \subset \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$.

(iii) $\left(\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} + \mathcal{C}\right) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$.

(iv) $\delta\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$, *for all $\delta \in \mathcal{I}_\Delta$.*

(v) *The set $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$ is convex.*

Proof of Lemma A.1. *Part (i)* Let $g \in \mathcal{G}$. By (8), any metatechnology $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ satisfies axiom IGT and, therefore, includes its gth group technology, i.e., $\mathcal{T}^g \subseteq \mathcal{T}^{\mathcal{G}}$. Then, the intersection of all metatechnologies $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ includes the intersection of all their corresponding gth group technologies, i.e., $\bigcap_{\substack{\mathcal{T}^g \sqsubseteq \mathcal{T}^{\mathcal{G}} \\ \mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}}} \mathcal{T}^g \subseteq \bigcap_{\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}} \mathcal{T}^{\mathcal{G}}$. By conditions (9a) and (9b), this embedding results in $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$.

Part (ii) Let $g \in \mathcal{G}$. By (8), any metatechnology $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ satisfies axiom GIO, and therefore, its corresponding gth group technology $\mathcal{T}^g$ includes observations in $\mathcal{N}_g$, i.e., $\mathcal{N}_g \subset \mathcal{T}^g$. Consequently, the intersection of gth group technologies of all metatechnologies $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ includes $\mathcal{N}_g$, i.e., $\mathcal{N}_g \subset \bigcap_{\substack{\mathcal{T}^g \sqsubseteq \mathcal{T}^{\mathcal{G}} \\ \mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}}} \mathcal{T}^g$. By (9b), this embedding results in $\mathcal{N}_g \subset \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$.

Part (iii) Let $g \in \mathcal{G}$. By (8), any metatechnology $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ satisfies axiom WGSD, and therefore, $(\mathcal{T}^g + \mathcal{C}) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}^g$. Then intersecting the corresponding left and right hand sides of this embedding gives

$$\bigcap_{\substack{\mathcal{T}^g \sqsubseteq \mathcal{T}^{\mathcal{G}} \\ \mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}}} (\mathcal{T}^g + \mathcal{C}) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \bigcap_{\substack{\mathcal{T}^g \sqsubseteq \mathcal{T}^{\mathcal{G}} \\ \mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}}} \mathcal{T}^g.$$

It is straightforward to verify that $\bigcap_{\substack{\mathcal{T}^g \sqsubseteq \mathcal{T}^{\mathcal{G}} \\ \mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}}} (\mathcal{T}^g + \mathcal{C}) = \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} + \mathcal{C}$. Taking into account this equality, the above embedding results in $\left(\mathcal{T}^{g}_{\Delta\text{-WGC}} + \mathcal{C}\right) \cap \left(\mathbb{R}^m_+ \times \mathbb{R}^s_+\right) \subseteq \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$.

Part (iv) Let $g \in \mathcal{G}$. By (8), any metatechnology $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ satisfies axiom Δ-WGRS and, therefore, the embedding $\delta \mathcal{T}^{g} \subseteq \mathcal{T}^{g}$ is true for all $\delta \in \mathcal{I}_{\Delta}$. Taking into account condition (9b), it follows that $\delta \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$, for all $\delta \in \mathcal{I}_{\Delta}$.

Part (v) Let $g \in \mathcal{G}$. By (8), any metatechnology $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ satisfies axiom WGC and, therefore, its corresponding gth group technology is convex. Consequently, the intersection of gth group technologies of all metatechnologies $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$ is convex. By (9b), it follows that $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$ is convex. $\square$

Proof of Theorem 4.1. Denote $\mathcal{U}$ the set on the right-hand side of equality (10). Consider $\mathcal{U}$ as a metatechnology whose group technologies are $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-WGC}}$, $g \in \mathcal{G}$. Taking into account (8), it follows from Lemma A.1 that $\mathcal{U} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. Then, condition (9a) in Definition 4.1 implies that $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}} \subseteq \mathcal{U}$.

Conversely, let $(\mathbf{x}, \mathbf{y}) \in \mathcal{U}$. Then, there exists a $g' \in \mathcal{G}$ such that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{g'\text{-TD}}_{\Delta\text{-WGC}}$. By condition (9b) in Definition 4.1, it follows that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{\mathcal{G}}$ for all $\mathcal{T}^{\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. Taking into account condition (9a) in Definition 4.1, it follows that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$. Therefore, $\mathcal{U} \subseteq \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$. $\square$

Proof of Theorem 4.2. By Lemma A.1, metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ satisfies axioms IGT, GIO, WGSD, Δ-WGRS and WGC, and therefore $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. Then, from condition (9a) in Definition 4.1, it follows that metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ satisfies the MEP in terms of the stated axioms. $\square$

Proof of Theorem 4.3. *Part (i)* Consider any $g' \in \mathcal{G}$. Then, the set $\mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$ is pre-defined in (6) as the g'th BU group technology: $\mathcal{T}^{g'\text{-BU}}_{\Delta\text{-WGC}} = \mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$. Thus, it suffices to prove that $\mathcal{T}^{g'\text{-TD}}_{\Delta\text{-WGC}} = \mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$.

By parts (ii)–(v) of Lemma A.1, the set $\mathcal{T}^{g'\text{-TD}}_{\Delta\text{-WGC}}$ includes observations in $\mathcal{N}_{g'}$ and satisfies axioms SD, Δ-RS and C. Because the Δ-C technology $\mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$ defined over the observations in $\mathcal{N}_{g'}$ is the smallest technology that encompasses these four properties, it follows that $\mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}} \subseteq \mathcal{T}^{g'\text{-TD}}_{\Delta\text{-WGC}}$. Conversely, consider the set $\bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}$ as a metatechnology whose group technologies are $\mathcal{T}^{g\text{-min}}_{\Delta\text{-C}}$, $g \in \mathcal{G}$. Then, this metatechnology satisfies axioms IGT, GIO, WGSD, Δ-WGRS and WGC. By (8), it follows that $\bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-min}}_{\Delta\text{-C}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. Then, the definition (9b) of the g'th TD group technology implies that $\mathcal{T}^{g'\text{-TD}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$. Therefore, $\mathcal{T}^{g'\text{-TD}}_{\Delta\text{-WGC}} = \mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$.

Part (ii) Taking into account statement (i) of the theorem, statement (ii) follows from the traditional BU definition (7) and the new TD definition (10) of the Δ-WGC metatechnology. $\square$

Proof of Theorem 4.4. *Part (i)* By Proposition 1 in Briec et al. (2000), any CRS technology is the union of its NIRS and NDRS counterparts. It follows that the equality $\mathcal{T}^{g\text{-min}}_{\text{C-WGC}} = \mathcal{T}^{g\text{-min}}_{\text{NI-WGC}} \cup \mathcal{T}^{g\text{-min}}_{\text{ND-WGC}}$ is true for all $g \in \mathcal{G}$. Taking the union of both sides of this equality over g and then using statement (ii) of Theorem 4.3, we obtain the equality $\mathcal{T}^{\mathcal{G}}_{\text{C-WGC}} = \mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}} \cup \mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$.

Part (ii) By Proposition 1 in Briec et al. (2000), the equality $\mathcal{T}^{g\text{-min}}_{\text{V-WGC}} = \mathcal{T}^{g\text{-min}}_{\text{NI-WGC}} \cap \mathcal{T}^{g\text{-min}}_{\text{ND-WGC}}$ is true for all $g \in \mathcal{G}$. Taking the union of both sides of this equality over g, we obtain the equality $\bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-min}}_{\text{V-C}} = \bigcup_{g \in \mathcal{G}} \big(\mathcal{T}^{g\text{-min}}_{\text{NI-C}} \cap \mathcal{T}^{g\text{-min}}_{\text{ND-C}}\big)$. Additionally, the set on the right-hand side of this equality is a subset of $\big(\bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-min}}_{\text{NI-C}}\big) \cap \big(\bigcup_{g \in \mathcal{G}} \mathcal{T}^{g\text{-min}}_{\text{ND-C}}\big)$. Therefore, using statement (i) of Theorem 4.3, we obtain the embedding $\mathcal{T}^{\mathcal{G}}_{\text{V-WGC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\text{NI-WGC}} \cap \mathcal{T}^{\mathcal{G}}_{\text{ND-WGC}}$. $\square$

Proof of Theorem 4.5. Let $\mathcal{V}$ denote the set on the right-hand side of (11). We need to prove that $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \mathcal{V}$. Let $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. By Theorem 4.3, there exists a $g' \in \mathcal{G}$ such that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$.

By (2), it follows that there exist a scalar $\delta \in \mathcal{I}_\Delta$ and a vector $\boldsymbol{\lambda}^{g'}$ such that

$$\sum_{j \in \mathcal{J}_{g'}} \lambda_j^{g'} \mathbf{x}_j \leq \mathbf{x}, \sum_{j \in \mathcal{J}_{g'}} \lambda_j^{g'} \mathbf{y}_j \geq \mathbf{y}, \mathbf{1}^\top \boldsymbol{\lambda}^{g'} = \delta, \boldsymbol{\lambda}^{g'} \geq \mathbf{0}. \tag{A.1}$$

Define $\hat{\gamma}_{g'} = 1$, $\hat{\boldsymbol{\lambda}}^{g'} = \boldsymbol{\lambda}^{g'}$, and $\hat{\gamma}_g = 0$, $\hat{\boldsymbol{\lambda}}^g = \mathbf{0}$ for all $g \neq g'$, $g \in \mathcal{G}$. Then, $(\mathbf{x}, \mathbf{y})$ satisfies (11) with the scalar δ and the vectors $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\gamma}}$. Therefore, $(\mathbf{x}, \mathbf{y}) \in \mathcal{V}$, and $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} \subseteq \mathcal{V}$.

Conversely, let $(\mathbf{x}, \mathbf{y}) \in \mathcal{V}$. Then, $(\mathbf{x}, \mathbf{y})$ satisfies (11) with some scalar $\delta \in \mathcal{I}_\Delta$ and some vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$. Because the sum of binary variables γ_g is equal to one, there exists exactly one $g' \in \mathcal{G}$ such that $\gamma_{g'} = 1$, and $\gamma_g = 0$ for all $g \neq g'$, $g \in \mathcal{G}$. Then, the set $\mathcal{V}$ becomes equal to $\mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$, and thus $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$. By Theorem 4.3, it follows that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. Therefore, $\mathcal{V} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}$. □

Proof of Theorem 4.6. The proof of this theorem is similar to the proof of Theorem 4.5 and it is therefore omitted. □

Lemma A.2. *The optimal value $\tau_o^\Delta(\mathbf{d})$ of program (15) is equal to the optimal value of the following nonlinear program:*

$$\begin{aligned} \max \quad & \tau \\ \text{s.t.} \quad & (15\text{b}) - (15\text{e}), \\ & \boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\gamma} \geq \mathbf{0}, \tau \text{ sign free.} \end{aligned} \tag{A.2}$$

Proof of Lemma A.2. Let $(\tau^*, \boldsymbol{\lambda}^*, \boldsymbol{\gamma}^*)$ be an optimal solution to program (A.2). Because this solution satisfies the constraints of program (A.2), we have $\mathbf{1}^\top \boldsymbol{\gamma}^* = 1$. This guarantees the existence of a $g' \in \mathcal{G}$ such that $\gamma^*_{g'} > 0$. Corresponding to such g', the first three group-wise constraints of program (A.2) can be equivalently stated as follows:

$$\sum_{j \in \mathcal{J}_{g'}} \frac{\lambda_j^{g'*}}{\gamma^*_{g'}} \mathbf{x}_j \leq \mathbf{x}_o - \tau^* \mathbf{d}_x, \quad \sum_{j \in \mathcal{J}_{g'}} \frac{\lambda_j^{g'*}}{\gamma^*_{g'}} \mathbf{y}_j \geq \mathbf{y}_o + \tau^* \mathbf{d}_y, \quad \mathbf{1}^\top \big(\frac{1}{\gamma^*_{g'}} \boldsymbol{\lambda}^{g'*}\big) - 1 \in \frac{1}{\gamma^*_{g'}} \mathcal{D}_\Delta = \mathcal{D}_\Delta. \tag{A.3}$$

Define $\hat{\tau} = \tau^*$, $\hat{\boldsymbol{\lambda}}^{g'} = \frac{1}{\gamma^*_{g'}} \boldsymbol{\lambda}^{g'*}$, $\hat{\gamma}_{g'} = 1$, and $\hat{\boldsymbol{\lambda}}^g = \mathbf{0}$, $\hat{\gamma}_g = 0$ for all $g \in \mathcal{G} \backslash \{g'\}$. Then, it follows from (A.3) that $\big(\hat{\tau}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\gamma}}\big)$ is a feasible solution to program (15). Therefore, $\tau^* \leq \tau_o^\Delta(\mathbf{d})$.

Conversely, because any feasible solution of program (A.2) satisfies the constraints $\mathbf{1}^\top \boldsymbol{\gamma} = 1$ and $\boldsymbol{\gamma} \geq \mathbf{0}$, we have $\boldsymbol{\gamma} \leq \mathbf{1}$. This indicates that program (A.2) results from relaxing the binary constraints of program (15). Therefore, $\tau^* \geq \tau_o^\Delta(\mathbf{d})$, and $\tau^* = \tau_o^\Delta(\mathbf{d})$. □

Proof of Theorem 4.7. Let $(\tau^*, \boldsymbol{\lambda}^*, \boldsymbol{\gamma}^*)$ be an optimal solution to the nonlinear program (A.2). Then, Lemma A.2 implies that $\tau^* = \tau_o^\Delta(\mathbf{d})$. Thus, to prove the equality $\tau_o^\Delta(\mathbf{d}) = \hat{\tau}_o^\Delta(\mathbf{d})$, it suffices to show that $\tau^* = \hat{\tau}_o^\Delta(\mathbf{d})$. Define $\hat{\boldsymbol{\tau}} = \tau^* \boldsymbol{\gamma}^*$, $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^*$ and $\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma}^*$. Then, it follows from the constraint $\mathbf{1}^\top \boldsymbol{\gamma}^* = 1$ that $\mathbf{1}^\top \hat{\boldsymbol{\tau}} = \tau^*$. It is straightforward to verify that $\big(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\gamma}}\big)$ is a feasible solution to program (16). Therefore, $\tau^* \leq \hat{\tau}_o^\Delta(\mathbf{d})$.

Conversely, let $(\boldsymbol{\tau}^{**}, \boldsymbol{\lambda}^{**}, \boldsymbol{\gamma}^{**})$ be an optimal solution to the linear program (16). Because $(\boldsymbol{\lambda}^{**}, \boldsymbol{\gamma}^{**})$ is a feasible solution of program (16), we have $\mathbf{1}^\top \boldsymbol{\gamma}^{**} = 1$. This guarantees the existence of a $g' \in \mathcal{G}$

such that $\gamma_{g'}^* > 0$. Corresponding to such g', the first three group-wise constraints of program (16) can be equivalently stated as follows:

$$\sum_{j\in\mathcal{J}_{g'}} \frac{\lambda_j^{g'**}}{\gamma_{g'}^{**}}\mathbf{x}_j \le \mathbf{x}_o - \frac{\tau_{g'}^{**}}{\gamma_{g'}^{**}}\mathbf{d}_x, \quad \sum_{j\in\mathcal{J}_{g'}} \frac{\lambda_j^{g'**}}{\gamma_{g'}^{**}}\mathbf{y}_j \ge \mathbf{y}_o + \frac{\tau_{g'}^{**}}{\gamma_{g'}^{**}}\mathbf{d}_y, \quad \mathbf{1}^\top\big(\frac{1}{\gamma_{g'}^{**}}\boldsymbol{\lambda}^{g'**}\big) - 1 \in \frac{1}{\gamma_{g'}^{**}}\mathcal{D}_\Delta = \mathcal{D}_\Delta. \tag{A.4}$$

Define $\hat{\tau} = \frac{\tau_{g'}^{**}}{\gamma_{g'}^{**}}$, $\hat{\boldsymbol{\lambda}}^{g'} = \frac{1}{\gamma_{g'}^{**}}\boldsymbol{\lambda}^{g'**}$, $\hat{\gamma}_{g'} = 1$, and $\hat{\boldsymbol{\lambda}}^{g} = \mathbf{0}$, $\hat{\gamma}_g = 0$ for all $g \in \mathcal{G}\backslash\{g'\}$. Then, it follows from (A.4) that $(\hat{\tau}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\gamma}})$ is a feasible solution to program (A.2). By the optimality of $(\tau^*, \boldsymbol{\lambda}^*, \boldsymbol{\gamma}^*)$, it follows that $\tau^* \ge \hat{\tau}$ and, consequently, $\gamma_{g'}^{**}\tau^* \ge \tau_{g'}^{**}$. Then, summing up both sides of these inequalities over g' implies that $\tau^* \ge \hat{\tau}_o^\Delta(\mathbf{d})$. Therefore, $\tau^* = \hat{\tau}_o^\Delta(\mathbf{d})$. □

Lemma A.3. *The optimal value $\varepsilon_o(\mathbf{d})$ of program (23) is equal to the optimal value of the following nonlinear program:*

$$\begin{aligned} \max\quad & \frac{1}{\mathbf{1}^\top\bar{\mathbf{d}}}\Big(\sum_{i\in\mathcal{I}^+}\bar{d}_{xi}\beta_{xi} + \sum_{r\in\mathcal{O}^+}\bar{d}_{yr}\beta_{yr}\Big) \\ \text{s.t.}\quad & (23\text{b}) - (23\text{e}), \\ & \boldsymbol{\lambda} \ge \mathbf{0},\ \boldsymbol{\gamma} \ge \mathbf{0},\ \boldsymbol{\beta} \text{ sign free.} \end{aligned} \tag{A.5}$$

Proof of Lemma A.3. Let $(\boldsymbol{\lambda}^*, \boldsymbol{\gamma}^*, \boldsymbol{\beta}^*)$ be an optimal solution to program (A.5). Because $\mathbf{1}^\top\boldsymbol{\gamma}^* = 1$, there exists a $g' \in \mathcal{G}$ such that $\gamma_{g'}^* > 0$. Define $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}^*$, $\hat{\boldsymbol{\lambda}}^{g'} = \frac{1}{\gamma_{g'}^*}\boldsymbol{\lambda}^{g'*}$, $\hat{\gamma}_{g'} = 1$, and $\hat{\boldsymbol{\lambda}}^g = \mathbf{0}$, $\hat{\gamma}_g = 0$ for all $g \in \mathcal{G}\backslash\{g'\}$. Then, is straightforward to verify that $(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\gamma}})$ is a feasible solution to program (23). Therefore, $\frac{1}{\mathbf{1}^\top\bar{\mathbf{d}}}\big(\sum_{i\in\mathcal{I}^+}\bar{d}_{xi}\beta_{xi}^* + \sum_{r\in\mathcal{O}^+}\bar{d}_{yr}\beta_{yr}^*\big) \le \varepsilon_o(\mathbf{d})$.

Conversely, because any feasible solution of program (A.5) satisfies the constraints $\mathbf{1}^\top\boldsymbol{\gamma} = 1$ and $\boldsymbol{\gamma} \ge \mathbf{0}$, we have $\boldsymbol{\gamma} \le \mathbf{1}$. This indicates that program (A.5) results from relaxing the binary constraints of program (23). Therefore, $\frac{1}{\mathbf{1}^\top\bar{\mathbf{d}}}\big(\sum_{i\in\mathcal{I}^+}\bar{d}_{xi}\beta_{xi}^* + \sum_{r\in\mathcal{O}^+}\bar{d}_{yr}\beta_{yr}^*\big) \ge \varepsilon_o(\mathbf{d})$, and $\frac{1}{\mathbf{1}^\top\bar{\mathbf{d}}}\big(\sum_{i\in\mathcal{I}^+}\bar{d}_{xi}\beta_{xi}^* + \sum_{r\in\mathcal{O}^+}\bar{d}_{yr}\beta_{yr}^*\big) = \varepsilon_o(\mathbf{d})$. □

Proof of Theorem 4.8. By Lemma A.3, it suffices to show that programs (24) and (A.5) are equivalent. To prove this, let $g \in \mathcal{G}$. For any $i \in \mathcal{I}^+$ and any $r \in \mathcal{O}^+$, the first and second group-wise inequality constraints of program (A.5) hold at optimality as equalities. Therefore, the inequality sign of the stated constraint can be changed to an equality sign:

$$\sum_{j\in\mathcal{J}_g}\lambda_j^g x_{ij} = \gamma_g x_{io} - \gamma_g\,\beta_{xi}d_{xi}, \tag{A.6a}$$

$$\sum_{j\in\mathcal{J}_g}\lambda_j^g y_{rj} = \gamma_g y_{ro} + \gamma_g\,\beta_{yr}d_{yr}. \tag{A.6b}$$

Multiply both sides of equalities (A.6a) and (A.6b) by w_i and q_r, respectively:

$$\sum_{j\in\mathcal{J}_g}\lambda_j^g w_i x_{ij} = \gamma_g w_i x_{io} - \gamma_g\,\beta_{xi}\bar{d}_{xi}, \tag{A.7a}$$

$$\sum_{j\in\mathcal{J}_g}\lambda_j^g q_r y_{rj} = \gamma_g q_r y_{ro} + \gamma_g\,\beta_{yr}\bar{d}_{yr}. \tag{A.7b}$$

Sum up both sides of (A.7a) over $i \in \mathcal{I}^+$ and both sides of (A.7b) over $r \in \mathcal{O}^+$:

$$\sum_{j\in\mathcal{J}_g} \lambda_j^g \sum_{i\in\mathcal{I}^+} w_i x_{ij} = \gamma_g \sum_{i\in\mathcal{I}^+} w_i x_{io} - \gamma_g \sum_{i\in\mathcal{I}^+} \bar{d}_{xi}\beta_{xi}, \tag{A.8a}$$

$$\sum_{j\in\mathcal{J}_g} \lambda_j^g \sum_{r\in\mathcal{O}^+} q_r y_{rj} = \gamma_g \sum_{r\in\mathcal{O}^+} q_r y_{ro} + \gamma_g \sum_{r\in\mathcal{O}^+} \bar{d}_{yr}\beta_{yr}. \tag{A.8b}$$

Then, the following equality follows from subtracting (A.8a) from (A.8b):

$$\sum_{j\in\mathcal{J}_g} \lambda_j^g E_j = \gamma_g E_o + \gamma_g \big(\sum_{i\in\mathcal{I}^+} \bar{d}_{xi}\beta_{xi} + \sum_{r\in\mathcal{O}^+} \bar{d}_{yr}\beta_{yr} \big). \tag{A.9}$$

By summing up both sides of (A.9) over $g \in \mathcal{G}$, the objective function of program (A.5) is stated as follows:

$$\sum_{i\in\mathcal{I}^+} \bar{d}_{xi}\beta_{xi} + \sum_{r\in\mathcal{O}^+} \bar{d}_{yr}\beta_{yr} = \sum_{g\in\mathcal{G}} \sum_{j\in\mathcal{J}_g} \lambda_j^g E_j - E_o. \tag{A.10}$$

Based on (A.10), we replace the objective function of program (A.5) with its equivalent expression and remove the constraints by which this expression is obtained. Then, program (A.5) is converted to program (24). □

Proof of Corollary 4.2. Let $\mathcal{I}^+ = \mathcal{I}$ and $\mathcal{O}^+ = \mathcal{O}$. Then, taking into account Theorem 4.8, program (24) is converted into the following linear program:

$$\varepsilon_o(\mathbf{d}) = \max \ \frac{1}{\mathbf{1}^\top \bar{\mathbf{d}}} \Big(\sum_{g\in\mathcal{G}} \sum_{j\in\mathcal{J}_g} \lambda_j^g \pi_j - \pi_o \Big) \tag{A.11a}$$

$$\text{s.t.} \quad \mathbf{1}^\top \boldsymbol{\lambda}^g = \gamma_g, \quad g \in \mathcal{G}, \tag{A.11b}$$

$$\mathbf{1}^\top \boldsymbol{\gamma} = 1, \tag{A.11c}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \ \boldsymbol{\gamma} \geq \mathbf{0}. \tag{A.11d}$$

By the constraints (A.11b) and (A.11c), we have $\sum_{g\in\mathcal{G}} \sum_{j\in\mathcal{J}_g} \lambda_j^g = 1$. It follows that the maximum value of $\sum_{g\in\mathcal{G}} \sum_{j\in\mathcal{J}_g} \lambda_j^g \pi_j$ is given by $\max_{g\in\mathcal{G}} \max_{j\in\mathcal{J}_g} \pi_j$. Therefore, the equality (25) is true. □

Lemma A.4. *For each $g \in \mathcal{G}$, the following statements are true:*

(i) $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$.

(ii) $\mathcal{N}_g \subset \mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}}$.

(iii) $\Big(\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}} + \mathcal{C}\Big) \cap \big(\mathbb{R}^m_+ \times \mathbb{R}^s_+\big) \subseteq \mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}}$.

(iv) $\delta\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}}$*, for all* $\delta \in \mathcal{I}_\Delta$.

Proof of Lemma A.4. The proofs of parts (i)–(iv) are similar to their corresponding parts in Lemma A.1 and are therefore omitted. □

Proof of Theorem 5.1. Let $\mathcal{K}$ denote the set on the right hand-side of (32). Consider $\mathcal{K}$ as a metatechnology whose group technologies are $\mathcal{T}^{g\text{-TD}}_{\Delta\text{-NC}}$, $g \in \mathcal{G}$. Taking into account (30), it follows that $\mathcal{K} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}$. Then, condition (9a) in Definition 5.1 implies that $\mathcal{K} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}$. Then, condition (31a) in Definition 5.1 implies that $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}} \subseteq \mathcal{K}$.

Conversely, let $(\mathbf{x}, \mathbf{y}) \in \mathcal{K}$. Then, there exists a $g' \in \mathcal{G}$ such that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{g'\text{-TD}}_{\Delta\text{-NC}}$. By part (i) of Lemma A.4, it follows that $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$. Therefore, $\mathcal{K} \subseteq \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$. □

Proof of Theorem 5.2. By Lemma A.4, metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-WGC}}$ satisfies axioms IGT, GIO, WGSD and Δ-WGRS, and thus $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$. Then, from condition (31a) in Definition 5.1, it follows that metatechnology $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}}$ satisfies the MEP in terms of the stated axioms. □

Proof of Theorem 5.3. *Part (i)* Consider any $g' \in \mathcal{G}$. Then, the set $\mathcal{T}^{g'\text{-min}}_{\Delta\text{-NC}}$ is pre-defined in (27) as the g'th BU group technology: $\mathcal{T}^{g'\text{-BU}}_{\Delta\text{-NC}} = \mathcal{T}^{g'\text{-min}}_{\Delta\text{-NC}}$. Thus, it suffices to prove that $\mathcal{T}^{g'\text{-TD}}_{\Delta\text{-NC}} = \mathcal{T}^{g'\text{-min}}_{\Delta\text{-NC}}$.

By statements (ii)–(iv) of Lemma A.4, the set $\mathcal{T}^{g'\text{-TD}}_{\Delta\text{-NC}}$ includes observations in $\mathcal{N}_{g'}$ and satisfies axioms SD and Δ-RS. Because the Δ-NC technology $\mathcal{T}^{g'\text{-min}}_{\Delta\text{-C}}$ defined over the observations in $\mathcal{N}_{g'}$ is the smallest technology that encompasses these three properties, we have $\mathcal{T}^{g'\text{-min}}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{g'\text{-TD}}_{\Delta\text{-NC}}$.

Conversely, consider the set $\bigcup_{g\in\mathcal{G}} \mathcal{T}^{g\text{-min}}_{\Delta\text{-NC}}$ as a metatechnology whose group technologies are $\mathcal{T}^{g\text{-min}}_{\Delta\text{-NC}}$, $g \in \mathcal{G}$. Then, this metatechnology satisfies axioms IGT, GIO, WGSD and Δ-WGRS, and therefore, $\bigcup_{g\in\mathcal{G}} \mathcal{T}^{g\text{-min}}_{\Delta\text{-NC}} \in \mathbb{T}^{\mathcal{G}}_{\Delta\text{-NC}}$. Then, the definition (31b) of the g'th TD group technology implies that $\mathcal{T}^{g'\text{-TD}}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{g'\text{-min}}_{\Delta\text{-NC}}$.

Part (ii) Taking into account statement (i) of the theorem, the equality $\mathcal{T}^{\mathcal{G}\text{-TD}}_{\Delta\text{-NC}} = \mathcal{T}^{\mathcal{G}\text{-BU}}_{\Delta\text{-NC}}$ follows from the traditional BU definition (28) and the new TD definition (32) of the Δ-NC metatechnology. Thus, the proof is completed by incorporating equality (29). □

Proof of Corollary 5.1. Let $\mathcal{N}_j = \{(\mathbf{x}_j, \mathbf{y}_j)\}$ for all $j \in \mathcal{J}$. For each $j \in \mathcal{J}$, let $\mathcal{T}^{j\text{-min}}_{\Delta\text{-C}}$ denote the Δ-C technology generated by the jth observation. From statement (ii) of Theorem 4.3, it follows that $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} = \bigcup_{j\in\mathcal{J}} \mathcal{T}^{j\text{-min}}_{\Delta\text{-C}}$. By (2), let us restate technology $\mathcal{T}^{j\text{-min}}_{\Delta\text{-C}}$ as $\mathcal{T}^{j\text{-min}}_{\Delta\text{-C}} = \{(\delta\mathbf{x}, \delta\mathbf{y}) \mid \mathbf{x} \geq \mathbf{x}_j,\ \mathbf{0} \leq \mathbf{y} \leq \mathbf{y}_j,\ \delta \in \mathcal{I}_\Delta\}$. Then, taking into account this statement, the proof is completed by equations (2) and (3) in Briec et al. (2004, p. 164). □

Proof of Theorem 5.4. From Kerstens and Vanden Eeckaut (1999), the equality $\mathcal{T}^{g\text{-min}}_{\text{C-NC}} = \mathcal{T}^{g\text{-min}}_{\text{NI-NC}} \cup \mathcal{T}^{g\text{-min}}_{\text{ND-NC}}$ is true for all $g \in \mathcal{G}$. Taking the union of both sides of this equality over g and then using Theorem 5.3, we obtain the equality $\mathcal{T}^{\mathcal{G}}_{\text{C-NC}} = \mathcal{T}^{\mathcal{G}}_{\text{NI-NC}} \cup \mathcal{T}^{\mathcal{G}}_{\text{ND-NC}}$.

Similarly, from Kerstens and Vanden Eeckaut (1999), the embedding $\mathcal{T}^{g\text{-min}}_{\text{V-NC}} \subseteq \mathcal{T}^{g\text{-min}}_{\text{NI-NC}} \cap \mathcal{T}^{g\text{-min}}_{\text{ND-NC}}$ is true for all $g \in \mathcal{G}$. Taking the union of both sides of these embeddings over g, we obtain the embedding $\bigcup_{g\in\mathcal{G}} \mathcal{T}^{g\text{-min}}_{\text{V-NC}} \subseteq \bigcup_{g\in\mathcal{G}} \big(\mathcal{T}^{g\text{-min}}_{\text{NI-NC}} \cap \mathcal{T}^{g\text{-min}}_{\text{ND-NC}}\big)$. Additionally, the set on the right-hand side of this embedding is a subset of $\big(\bigcup_{g\in\mathcal{G}} \mathcal{T}^{g\text{-min}}_{\text{NI-NC}}\big) \cap \big(\bigcup_{g\in\mathcal{G}} \mathcal{T}^{g\text{-min}}_{\text{ND-NC}}\big)$. Therefore, using statement (i) of Theorem 5.3, we obtain the embedding $\mathcal{T}^{\mathcal{G}}_{\text{V-NC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\text{NI-NC}} \cap \mathcal{T}^{\mathcal{G}}_{\text{ND-NC}}$. □

Proof of Theorem 5.5. By Corollary 5.1, the Δ-WGC and Δ-NC metatechnologies are identical in the special case that $\mathcal{J}_j = \{j\}$ for all $j \in \mathcal{J}$. Using this fact, statement (13) of the Δ-WGC metatechnology is converted to the statement (35) of the Δ-NC metatechnology. □

Proof of Theorem 5.6. By Corollary 5.1, the Δ-WGC and Δ-NC metatechnologies are identical in the special case that $\mathcal{J}_j = \{j\}$ for all $j \in \mathcal{J}$. Using this fact, the linear program (24) is converted to program (38). $\square$

Proof of Corollary 5.2. Similar to the proof of Corollary 4.2, the proof of this corollary follows from implementing the conditions $\mathcal{I}^+ = \mathcal{I}$ and $\mathcal{O}^+ = \mathcal{O}$ on the linear program (38). $\square$

Proof of Theorem 6.1. Consider the Δ-C technology $\mathcal{T}^{\min}_{\Delta\text{-C}}$ as a metatechnology whose group technologies are $\mathcal{T}^{g\text{-}\min}_{\Delta\text{-C}}$, $g \in \mathcal{G}$. Then, it is straightforward to verify that this metatechnology satisfies axioms IGT, GIO, WGSD, Δ-WGRS, WGC and BGC. Since pseudo-metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ is the smallest metatechnology that satisfies these axioms, we have $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}} \subseteq \mathcal{T}^{\min}_{\Delta\text{-C}}$.

By part (ii) of Theorem 5.3, we have $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}} = \mathcal{T}^{\min}_{\Delta\text{-NC}}$. By (33) and (41), it thus follows that $\mathcal{T}^{\min}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$. Since metatechnology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ is convex, it includes the convex hull of $\mathcal{T}^{\min}_{\Delta\text{-NC}}$, which is equal to $\mathcal{T}^{\min}_{\Delta\text{-C}}$ by (3). Therefore, $\mathcal{T}^{\min}_{\Delta\text{-C}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$. $\square$

Proof of Theorem 6.3. By part (ii) of Theorem 5.3, we have $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-NC}} = \mathcal{T}^{\min}_{\Delta\text{-NC}}$. By (33) and (41), it follows that $\mathcal{T}^{\min}_{\Delta\text{-NC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}} \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$. Since meta-technology $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$ is convex, it results by (3) that $\mathcal{T}^{\min}_{\Delta\text{-C}} \subseteq \operatorname{conv}\left(\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}\right) \subseteq \mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}}$. By Theorem 6.1, we therefore have $\mathcal{T}^{\mathcal{G}}_{\Delta\text{-C}} = \operatorname{conv}\left(\mathcal{T}^{\mathcal{G}}_{\Delta\text{-WGC}}\right)$. $\square$